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OPTIMAL ECONOMIC GROWTH AND TURNPIKE THEOREMS*

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1. Introduction

We will be concerned with the long term tendencies of paths of capital accumulation that maximize, in some sense, a utility sum for society over an unbounded time span. However, the structure of the problem is characteristic of all economizing over time whether on the social scale, or the scale of the individual or the firm. The mathematical methods ~~that~~ will be used ~~are closely~~ allied to the old mathematical discipline, calculus of variations. This is due to the fact that both are concerned with the theory of maxima of functions where the number of variables is infinite. However, the infinity that is essential to the problem of capital accumulation is not the consequence of treating time as a continuum but of the use of an infinite horizon. For this reason we will avoid many of the complications of the calculus of variations by using discrete time as the independent variable. On the other hand, the nature of our problem requires that we go beyond the traditional approach to consider paths of capital stocks that meet the boundaries of the regions within which they must lie given the conditions of the problem, in particular, the requirement that the capital stocks be non-negative. Of course, this is one of principal modern innovations in the theory of maximization from the work of writers such as Kuhn and Tucker [1951], Bellman [1957], and Pontryagin [1962].

A crucial condition for the maximum to be achieved, whether as a necessary condition or as one of the sufficient conditions, has been concavity of the maximand, at least locally at the maximal path. This is to be expected from the conditions for a maximum of a function of a finite number of variables. In the calculus of variations the convacity that is needed is provided by the conditions of Weierstrass and Lagrange (see Bliss [1925] for a classical reference or Hestenes [1966] for a modern reference). Moreover, when global results are sought, the concavity condition is assumed throughout a relevant region. This

is also to be expected from the theory with a finite number of variables. In our theory concavity of the utility function will always be assumed, even uniformly over a relevant region for the global maximum and over time. The utility is defined directly on the capital stocks at the beginning and the end of a standard period of time, and the concavity is with respect to these variables.

In most of the discussion the utility function will be allowed to depend on time, as in the standard theory of the calculus of variations. Also the function to be maximized will be the sum of utility functions for each period over the future. This is described as a separable utility function over the sequence of future capital stocks and corresponds to the integral of calculus of variations. Since the consumption of one period does influence the utility of later consumption, the separability assumption is not exact. However, the error is no doubt reduced by lengthening the period, though this may not be much help in an application of the theory. The treatment of utility in a period as dependent on initial and terminal stocks is not a restriction since the usual assumptions that make utility depend on consumption and consumption on production will imply that an equivalent utility depending on capital stocks exists.

The theory that I will present will cover both discounted and undiscounted utility. We will seek to determine the asymptotic behavior of maximal paths, in particular their tendency to cluster in the sufficiently distant future from whatever capital stocks they start. In models with stationary utility functions, perhaps subject to discounting, the clustering has been seen as convergence to a stationary path along which capital stocks are constant. This view is reinforced by the fact that in stationary models the existence of an optimal stationary path, which is, moreover, supported by prices, is easy to prove by special means which are not available for other maximal paths. Then this path and its prices can be used to establish the asymptotic convergence of other paths to it, with great ease in the undiscounted case. However, methods are now available from the work of Weitzman [1973] to derive the prices for other paths directly and the balanced path loses its distinguished role at least in the asymptotic theory where existence is assumed.

Our consideration will be confined to the deterministic model although using methods developed in this model analogous results have also been proved for the stochastic model in which the future is uncertain (Brock and Majumdar [1978]). Also most of the argument will assume concavity of the relevant functions without requiring differentiability or interior solutions. However, some consideration will be given to a differentiable case where the maximal path is assumed to lie in the interior of the region of definition of the utility function. These stronger assumptions are analogous to the assumptions used in the comparative statics of general equilibrium models of competitive economies. Here they will permit some comparative dynamics to be done. The assumptions are in some ways even stronger than those usual in classical calculus of variations. However, the methods that become available are very powerful in the discrete model and, so far as I know, have not been extended to models with continuous time as the independent variable.

The original context for the optimal growth model was the problem of the level of saving that would maximize a utility sum over future time for a population. This problem was solved by the Cambridge mathematician Frank Ramsey [1928] for a one good model, which may be thought of as an aggregated economy over an infinite future. The gambit used by him to handle the infinities involved is still useful today. However, the emphasis on asymptotic behavior for optimal paths came later in the multisector von Neumann model analysed by Dorfman, Samuelson, and Solow [1958]. They dealt with finite paths where the

objective was to maximize terminal stocks and their model contained two sectors. Since the model was stationary they could concentrate on the convergence of all optimal paths to a stationary optimal path. Later authors first extended the model to many sectors and then introduced a Ramsey style utility function on current consumption as the objective rather than terminal stocks. Also the horizon was extended to infinity. Von Weizsäcker [1965] generalized the objective function somewhat by defining the overtaking criterion in which attention is turned to finite sums and optimality is assigned to a path whose finite utility sums eventually dominate when it is compared with any alternative path from the same initial stocks. He also dealt with a model in which utility and production functions change over time, but he aggregated the economy to a single sector. We will deal essentially with the Weizsäcker model in a disaggregated form, which is natural when the analysis is directed to asymptotic behavior of paths.

Although the primary sources of the optimal growth model are aggregate savings programs and capital accumulation programs for an economy, the theorems and methods of the subject are finding applications in other areas with increasing frequency. For example, applications are being made to capital accumulation by the firm with adjustment costs Brock and Scheinkman [1978] and Scheinkman [1978], and to competitive markets with perfect foresight Brock [1974], or rational expectations Brock [1978]. In these models the social utility function is replaced by individuals' utility functions or by the profit functions of firms. Thus there is apparent a movement toward a general theory of economic dynamics in which asymptotic theorems and comparative dynamic theorems form the bulk of the results and where the analysis is largely derived from the optimal growth literature. It might be argued that our subject is best described as the study of economizing over time (see Intriligator [1971]).

2. The Basic Model

I will use a reduced form of the objective function in which utility is expressed as a function of the initial and terminal stocks of a period. The utility function is written $u_t(x,y)$ where x is the vector of capital stocks at time t-1 and y is the vector of capital stocks at time t. Then u_t is the utility derived from activities during the time period from t-1 to t, which I call the *tth* period. The reduced model is equivalent to the traditional extensive model where utility is expressed as a function $u_t(c)$ of the consumption vector in the *tth* period. The extensive model introduces a production correspondence $f_t(x)$ that expresses output, not just capital goods, as depending on initial capital stocks. However, so long as the utility functions of different periods are independent, it is a necessary condition for a maximal program that c be chosen from $f_t(x) - y$, where y is terminal stocks, to maximize u_t. Thus the models are not significantly different. It should be noted that the full commodity space in which $f_t(x)$ lies may include labor services and perishable goods dated by their times of use within the period.

We may allow the utility function u_t, as well as the space E_t of capital stock vectors at time t, to depend on t. Then u_t maps a set D_t contained in the nonnegative orthant of $E_{t-1} \times E_t$ into the real line, where E_{t-1} and E_t are Euclidean spaces of dimensions n_{t-1} and n_t respectively. We assume

(I) *The utility functions* $u_t(x,y)$ *are concave and closed for all* t. *The sets* D_t *are convex.*

(II) *If* $(x,y) \in D_t$ *and* $|x| < \xi < \infty$, *there is* $\zeta < \infty$ *such that* $u_t(x,y) < \zeta$ *and* $|y| < \zeta$.

Assumption I provides the concavity and convexity that are recurrent features of calculus of variations and other theories of maximization. By u_t is *closed* is meant that $(x^s,y^s) \to (x,y) \in$ boundary D_t implies $u_t(x^s,y^s) \to u_t(x,y)$ if

$(x,y) \in D_t$ and $u_t(x^s,y^s) \to -\infty$ otherwise. Since we are seeking global results, the assumptions on concavity and convexity are global. The boundedness assumptions in Assumption II are made to avoid trivial cases.

A sequence of capital stocks $\{k_t\}$, $t \in N$, is a path of accumulation if N is a set of consecutive integers and $(k_{t-1},k_t) \in D_t$ when t-1 and t are in N. The set N may be a finite or an infinite set.

We may note that the capital stocks are the state variables in the language of optimal control and there is no need to confine them to physical goods or things that can be appropriated as private or public property. For example, features of the environment, skills of workers, and mineral deposits may also be included. These comprise ways in which future utility possibilities may be influenced by present choices. In addition, the dependence of the utility functions on time may take account of trends in technology, tastes, and environment so far as they are independent of the choices. Of course, the interpretation of the state variables will depend on the particular problem at hand. My descriptions have been appropriate to the interpretation of u_t as a social utility function that is the objective of planning by the state.

In classical economics the concavity of the production correspondence that is part of the ground for Assumption I is often explained in terms of the independence and linearity of basic productive activities, at least to an approximation. However, when external effects are present so that different activities influence one another, this ground of concavity is jeopardized (Starrett [1972]). Also polluting substances in the environment are not allocated between activities the way capital goods are, so they do not fit into the paradigm of an allocation of stocks among independent, linear activities. These are important qualifications to the generality of our model.

3. The Objective Function

The objective function for a finite program from $t = 0$ to $t = T$ is $\sum_{t=1}^{T} u_t(x_{t-1}, y_t)$. If the sum exists, the objective function for an infinite program $\{k_t\}$ beginning at $t = 0$ is similarly $\sum_{t=1}^{\infty} u_t(k_{t-1}, k_t)$. However, the infinite sum may not exist and one of Ramsey's achievements was to show that this difficulty may be avoided in certain models with stationary utility functions by subtracting a constant from each term of the series to be summed. However, a more general method was introduced more recently by Weizsäcker and refined by Gale [1967] and Brock [1970]. In this approach the infinite sum is replaced by a comparison of finite sums. The new criterion is called the overtaking criterion.

Two definitions are made. The stronger definition characterizes an optimal path. We will say that a path $\{k_t\}$ *catches up to* a path $\{k_t'\}$ starting at the same time, if for any $\varepsilon > 0$ there is $T(\varepsilon)$ such that $\sum_{1}^{T}(u_t(k_{t-1}', k_t') - u_t(k_{t-1}, k_t)) < \varepsilon$ for all $T > T(\varepsilon)$. Then a path $\{k_t\}$ is *optimal* if it catches up to every alternative path from the same initial stocks. In other words, an optimal path is asymptotically as good as any other path from the same starting point when they are compared by means of their initial segments.

We will say that a path $\{k_t'\}$ *overtakes a path* $\{k_t\}$ starting at the same time, if there is $\varepsilon > 0$ and $T(\varepsilon)$ such that $\sum_{1}^{T}(u_t(k_{t-1}', k_t') - u_t(k_{t-1}, k_t)) > \varepsilon$ for all $T > T(\varepsilon)$. Then a path $\{k_t\}$ is *maximal* if there is no path from the same initial stocks that overtakes it. This says that a maximal path does not become permanently worse than some alternative path when they are compared by means of their initial segments.

4. Support Prices

We wish to allow for maximal paths that do not remain interior to the sets D_t at all times, or perhaps at any time. In these cases derivatives will not always exist for the utility functions along the path. For this reason it is convenient to introduce dual variables, which we call prices, as generalizations of derivatives. Then it is also possible to dispense with assumptions of differentiability in the interior of D_t as well. The existence of the appropriate prices for our purposes was proved by Weitzman [1973] in a somewhat more special context. However, his method can be adapted to our case (McKenzie [1976]). The corresponding theorem has been proved for the continuous time model by Benveniste and Scheinkman.

Consider a maximal path $\{k_t\}$, $t \in N$, where N is the set of non-negative integers. First, we normalize the utility function choosing the zeros of utility so that $u_t(k_{t-1}, k_t) = 0$ in every period. This is harmless since the choice of the zero level of utility has no effect on the comparison of paths. Next we define a value function $V_t(x)$ which values a capital stock at time t by the utility sums that can be got from it in the future. Following the example of Peleg and Zilcha [1977] in the stationary model, we set

$$(1) \quad V_t(x) = \sup(\liminf_{T\to\infty} \sum_{t+1}^{T} u_\tau(h_{\tau-1}, h_\tau))$$

over all paths $\{h_\tau\}$ with $h_t = x$. $V_t(x)$ is well defined when the right-hand side of (1) exists as a finite number or positive infinity. A little computation will show that the concavity of u_t and the convexity of D_t imply that $V_t(x)$ is concave and well defined on a convex set K_t. Since $V_t(k_t) = 0$ for all t, K_t is not empty. We may also note that $V_t(x)$ is well defined for any x for which there is a path $\{k_\tau'\}$ with $k_t' = x$ and $k_{t+n}' = k_{t+n}$.

Let P_t be the set of capital stocks y such that there is x with $(x,y) \in D_t$. P_t is the set of capital stocks that can be produced from some capital stocks held at time t-1. S is a *flat* in the Euclidean space E if there are vectors $y_i \in E$, $i \in N$, such that $z \in S$ is equivalent to $z = \sum_{i \in N} \alpha_i y_i$ for some numbers α_i such that $\sum_{i \in N} \alpha_i = 1$. For $t \geqq 1$, let S_t be the smallest flat in E_t that contains P_t and K_t. It is crucial to the derivation of support prices for $\{k_t\}$ to assume

(III) *Interior* $(P_t \cap K_t) \neq \phi$ *relative to* S_t, *for all* $t \geqq 1$. *Also* $k_0 \in$ *relative interior* K_0.

It is important to notice that Assumption III is not independent of the maximal path $\{k_t\}$, since the sets K_t depend on $V_t(x)$ which is defined after normalizing utility on $\{k_t\}$.

Since k_0 lies in the relative interior of K_0, given any $x \in K_0$, there is x' such that $k_0 = \alpha x + (1-\alpha)x'$ with $0 < \alpha < 1$ and $x' \in K_0$. Then, from the concavity of u_t and $V_0(k_0) = 0$, it follows that $V_0(x) < \infty$. But $V_0(x) < \infty$ and $(x,y) \in D_1$ implies $V_1(y) < \infty$. Since by Assumption III, y may be chosen in the interior of $P_1 \cap K_1$ relative to S_1, $V_1(x) < \infty$ for all $x \in K_1$. This argument can be continued to any $t > 0$, so $V_t(x) < \infty$ for $x \in K_t$ for all t. In interpreting the model it should be recalled that any goods not held at t = 0 may be omitted from E_0 and any goods that cannot be produced from k_0 after t periods may be omitted from E_t.

From the definition (1) of $V_t(x)$ it is clear that the principle of optimality holds and we may also write

$$(2) \quad V_t(x) = \sup(u_{t+1}(x,y) + V_{t+1}(y))$$

over all y such that $(x,y) \in D_{t+1}$ and $y \in K_{t+1}$. Make the induction assumption that there exists $p_t \in E_t$ (p_t may be 0) where $t \geqq 0$, such that

$$(3) \quad V_t(k_t) - p_t k_t \geqq V_t(x) - p_t x$$

over all $x \in K_t$. Let $x = k_t$ in (2). Then the sup is attained at $y = k_{t+1}$ by the maximality of $\{k_\tau\}$. The substitution of (2) in (3) gives

$$(4) \quad u_{t+1}(k_t,k_{t+1}) + V_{t+1}(k_{t+1}) - p_t k_t \geqq u_{t+1}(x,y) + V_{t+1}(y) - p_t x$$

for all $(x,y) \in D_{t+1}$ with $y \in K_{t+1}$. Denote the left side of (4), a given number, by v_{t+1}. Then

$$(5) \quad v_{t+1} - u_{t+1}(x,y) + p_t x \geqq V_{t+1}(y).$$

We define two sets for each $t \geqq 0$:

$A = \{(w,y) | y \in P_{t+1}$ and $w > v_{t+1} - u_{t+1}(x,y) + p_t x$ for some x with $(x,y) \in D_{t+1}\}$,

and

$B = \{(w,y) | y \in K_{t+1}$ and $w \leqq V_{t+1}(y)\}$.

By the existence of the path $\{k_t\}$, $P_{t+1} \cap K_{t+1} \neq \phi$. Thus A and B are not empty. A and B are disjoint by the inequality (5). They are also convex. Thus A and B may be separated by a hyperplane contained in $R \times E_{t+1}$, where R is the real line, defined by a vector $(\pi, -p_{t+1}) \neq 0$, where p_{t+1} lies in the linear subspace parallel to S_{t+1}. Then $\pi w - p_t y \geqq \pi w' - p_{t+1} y'$ for all $(w,y) \in A$ and $(w',y') \in B$. This situation is illustrated in Figure 1.

Using the definitions of w, w', and v_{t+1} and relation (4), the separation of A and B implies

$$(6) \quad \pi\{u_{t+1}(k_t,k_{t+1}) + V_{t+1}(k_{t+1}) - p_t k_t - u_{t+1}(x,y) + p_t x\} - p_{t+1} y$$
$$\geqq \pi V_{t+1}(y') - p_{t+1} y',$$

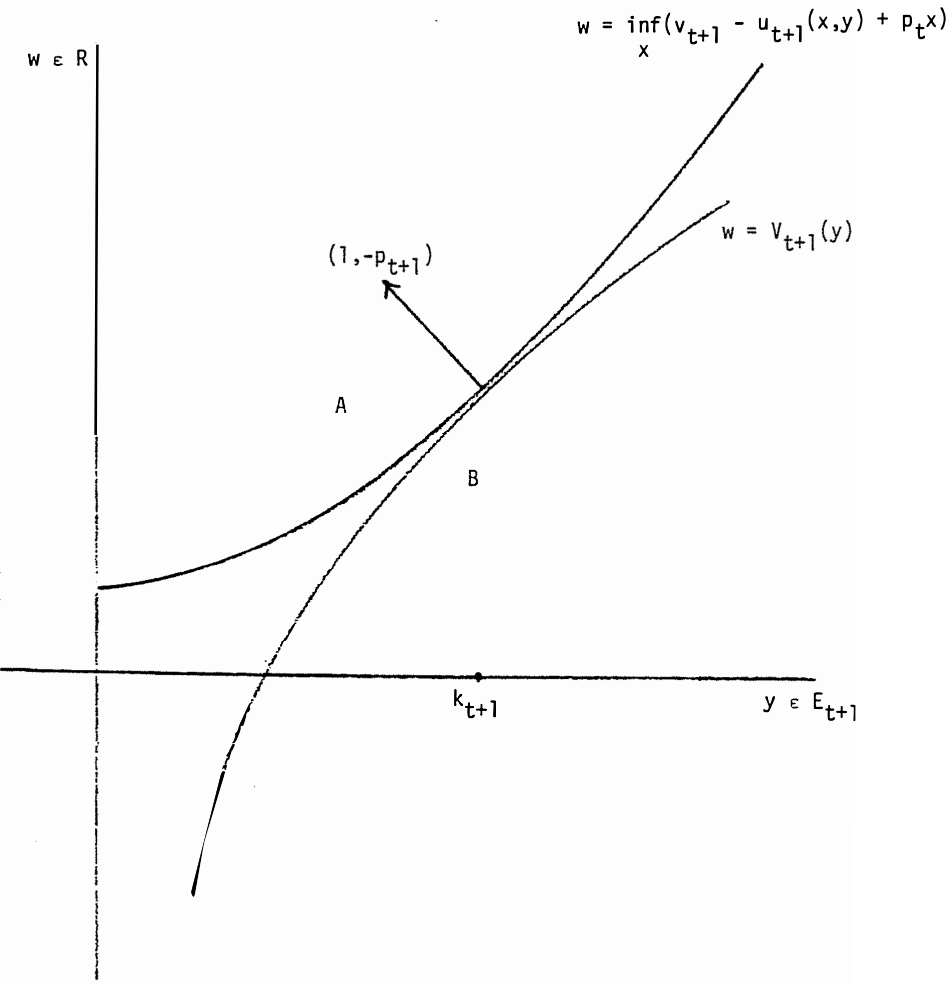

Figure 1

for any (x,y) such that $(x,y) \varepsilon D_t$ and any $y' \varepsilon K_{t+1}$. Put $x = k_t$, $y = k_{t+1}$ and (6) becomes

(7) $\pi V_{t+1}(k_{t+1}) - p_{t+1}k_{t+1} \geqq \pi V_{t+1}(y') - p_{t+1}y'$,

for all $y' \varepsilon K_{t+1}$. Put $y' = k_{t+1}$ and we obtain

(8) $\pi\{u_{t+1}(k_t,k_{t+1}) - p_t k_t\} + p_{t+1}k_{t+1} \geqq \pi\{u_{t+1}(x,y) - p_t x\} + p_{t+1}y$,

for all $(x,y) \varepsilon D_{t+1}$. If $\pi = 0$, (7) and (8) together imply $p_{t+1}k_{t+1} = p_{t+1}y$ over $P_{t+1} \cap K_{t+1}$. However, $P_{t+1} \cap K_{t+1}$ has an interior in S_{t+1} by Assumption III, and p_{t+1} lies in the linear subspace parallel to S_{t+1}. Therefore, $p_{t+1} = 0$ as well contradicting the requirement that $(\pi,-p_{t+1}) \neq 0$. Thus $\pi \neq 0$ and we may set $\pi = 1$.

The induction is begun by supporting the value function $V_0(y)$ at $k_0 \varepsilon K_0$ in $R \times E_0$. The concavity of $V_0(y)$ implies there is $(\pi,p_0) \neq 0$ such that $p_0 \varepsilon E_0$ and

(9) $\pi V_0(k_0) - p_0k_0 \geqq \pi V_0(x) - p_0x$,

for all $x \varepsilon K_0$. Choose p_0 in the linear subspace parallel to S_0 where S_0 is the smallest flat containing K_0. If $\pi = 0$, $p_0 \neq 0$ and $p_0(k_0-x) \leqq 0$ for all $x \varepsilon K_0$. Since $k_0 \varepsilon$ relative interior K_0, the inequality (9) is impossible, and $\pi \neq 0$. We may choose $\pi = 1$, as before.

We have proved that prices exist supporting maximal paths in the following sense:

Lemma 1. *Let $\{k_t\}$, $t = 0,1,\ldots$, be a maximal path of accumulation. If Assumptions* I, II, *and* III *are met, there exists a sequence of price vectors $p_t \varepsilon E_t$, $t = 0,1,\ldots$, which satisfy*

(11) $V_t(k_t) - p_t k_t \geqq V_t(y) - p_t y$, *for all* $y \in K_t$,

(12) $u_{t+1}(k_t, k_{t+1}) + p_{t+1}k_{t+1} - p_t k_t \geqq u_{t+1}(x,y) + p_{t+1}y - p_t x$, *for all* $(x,y) \in D_{t+1}$.

By (11) the prices support the value function. By (12) they support the utility function. These properties of the prices play crucial roles in the arguments leading to turnpike theorems for maximal paths when assumptions of differentiability of the utility function and interiority of paths are not made. The fact that the Weitzman prices support the value function implies that they are Malinvaud prices [1953], that is, k_t has minimal value at p_t over the set of capital stocks from which the subsequent utility stream can be obtained. This is obvious from (11), since $V_t(y) = V_t(k_t)$ implies $p_t k_t \leqq p_t y$. Of course, Malinvaud prices are defined for efficient paths rather than maximal paths and, in particular, summable utility is not needed. A path $\{k_t\}$, $t = 1,2,\ldots$, is said to be *efficient* if there is no path $\{k_t'\}$ with $k_0' = k_0$ such that $u_t(k_{t-1}', k_t') \geqq u_t(k_{t-1}, k_t)$ for all t with strict inequality for some t. It is clear that maximal paths must be efficient, but the contrary need not hold.

The converse of Lemma 1 is not true. However, a slight relaxation of the maximality conditions allows a converse result to be proved. The argument for Lemma 1 only requires that consecutive stocks along the path realize the supremum in equation (2), that is, for all t, it should be true that

$$V_t(k_t) = u_{t+1}(k_t, k_{t+1}) + V_{t+1}(k_{t+1}),$$

or equivalently that

(13) $V_0(k_0) = \sum_1^T u_t(k_{t-1}, k_t) + V_T(k_T)$, all $T \geqq 1$.

Then under Assumptions I, II, and III, the proof proceeds just as given. A path satisfying (13) may be called *agreeable*, since any "loss" from using an initial segment can be made arbitrarily small (a related idea may be found in Hammond [1975]). At any time T, given an arbitrary $\varepsilon > 0$, the initial segment of the agreeable path may be completed with a new choice of capital stocks beyond T, so that no path from the beginning can overtake the revised path by more than ε. As earlier, one path overtakes a second by ε if its finite sums eventually exceed those of the second path by ε at all subsequent times. If we call the revised path ε-maximal, the agreeable path can at any time be converted into an ε-maximal path where ε may be chosen arbitrarily small.

As before, choose the origin of the utility function in each period so that $u_t(k_{t-1}, k_t) = 0$ along the path. To prove the converse result, suppose a price sequence $\{p_t\}$, $t = 0,1,\ldots$, $p_t \in E_t$, exists such that (11) and (12) are satisfied for $\{k_t\}$. Assume that $\{k_t\}$ is not agreeable. Then for some T there is $\varepsilon > 0$ such that

(14) $V_0(k_0) \leqq \sum_1^T u_t(k_{t-1}, k_t) + V_t(k_T) + \varepsilon$.

But the definition of V_t implies there is some path $\{k_t'\}$, $t = 0,1,\ldots$, for which $k_0' = k_0$ and

(15) $V_0(k_0') \leqq \sum_1^T u_t(k_{t-1}', k_t') + V_T(k_T') + \varepsilon/2$.

Comparing (14) and (15) we derive

(16) $\sum_1^T u_t(k_{t-1}, k_t) + V_T(k_T) < \sum_1^T u_t(k_{t-1}', k_t') + V_T(k_T')$.

However, from (12) we have

$$(17)\quad \sum_1^T (u_t(k_{t-1},k_t) - u_t(k'_{t-1},k'_t)) \geqq p_T(k'_T-k_T) - p_0(k_0-k'_0),$$

and from (11)

$$(18)\quad V_T(k_T) - V_T(k'_T) \geqq p_T(k_T-k'_T).$$

Summing (17) and (18), and using $k_0 = k'_0$ gives

$$V_T(k_T) + \sum_1^T u_t(k_{t-1},k_t) \geqq V_T(k'_T) + \sum_1^T u_t(k'_{t-1},k'_t).$$

This contradicts (16) and proves the result. Thus we have

Theorem 1. *Under Assumptions* I, II, *and* III *a path is agreeable if and only if it may be price supported in the sense of Lemma* 1.

Notice that the Assumptions are not needed to prove that a price supported path is agreeable.

The cake eating example (Gale [1967], p. 4) is the classic example of a path that is agreeable but not maximal. The set D contains the pairs of numbers (x,y) such that $y \geqq 0$, $x \geqq 0$, and $y \leqq x$. Utility $u(x,y) = v(z)$ where $z = x-y \geqq 0$ and $v(z) = \log(1+z)$. The path $k_t = k_0$, all t, is agreeable but not maximal. Indeed, no maximal path exists though many agreeable paths exist. The path $k_t = k_0$ is supported by the prices $p_t = 1$, all t. The utility function is concave, but not strictly concave since $u(x,y) = u(x+z,y+z)$ for any $z \geqq -y$. Also $V_t(y) = y$, all t. Indeed, we may prove

Theorem 2. *If* u *is strictly concave, an agreeable path is unique.*

If a path is agreeable it satisfies (13). If there are two agreeable paths $\{k_t\}$ and $\{k'_t\}$, let T be the first time that $k_t \neq k'_t$. Then by strict concavity, the average of the right hand sides of (13) for k_t and k'_t is less than the value of the right hand side of (13) for the average of the two paths, k''_t, which is also feasible. That is,

$$(19)\quad V_0(k_0) < \sum_1^{T-1} u_t(k_{t-1},k_t) + u_T(k_{T-1},k''_T) + V_T(k''_T) + \varepsilon,$$

for some $\varepsilon > 0$. By definition of $V_T(k''_T)$ there is a path $\{\tilde{k}_t\}$ from $t = T$ such that $\sum_{T+1}^{\infty} u(\tilde{k}_{t-1},\tilde{k}_t) > V_T(k''_T) + \varepsilon/2$, where $\tilde{k}_T = k''_T$. Let $\tilde{k}_t = k_t$ for $t < T$. Then from (19)

$$V(k_0) < \sum_1^{\infty} u_t(\tilde{k}_{t-1},\tilde{k}_t),$$

in contradiction to the definition of $V(k_0)$. Thus there can be only one agreeable path. Under conditions to be explored in section 6, this path will be optimal. Theorem 2 was suggested by Hammond [1978].

It is sometimes valuable to know that capital values $p_t k_t$ are bounded as $t \to \infty$. Normalize utility on the agreeable path. A simple condition that guarantees boundedness of capital values is that $V_t(\alpha k_t)$ be bounded as $t \to \infty$, for any α with $|\alpha|$ sufficiently near 1. Consider

$$V_t(k_t) - p_t k_t \geqq V(\alpha k_t) - p_t(\alpha k_t), \text{ or}$$

$$(1-\alpha)p_t k_t \leqq V_t(k_t) - V_t(\alpha k_t).$$

Thus $p_t k_t$ is bounded above for $\alpha < 1$ if $V_t(\alpha k_t)$ is bounded below, since the normalization implies that $V_t(k_t) = 0$. Similarly, $\alpha > 1$ establishes a lower bound.

5. Optimal Paths

A useful basis for establishing the existence of optimal paths depends on having price supports for the utility function such that capital values are bounded along the path. In the case of certain stationary optimal paths stationary supports can be found by special arguments. Since capital values are then necessarily bounded, the stationary paths are optimal. Then value loss type arguments may be applied to prove that optimal paths originate from all capital stocks whose value functions are well defined relative to the stationary optimal path.

For the sake of the existence theorems we make three special assumptions, suggested by the methods of Weizsäcker,

(W1) *There is an infinite path* $\{k_t\}$, $t = 0,.,\ldots,$ *whose utility functions are supported by a price sequence* $\{p_t\}$ *in the sense of* (4.12).

(W2) *Lim sup* $p_t k_t = M < \infty$, *and if* $\{k'_t\}$ *is an infinite path with* $k'_0 = k_0$, *lim inf* $p_t k'_t > M' > -\infty$.

Let the *value loss* $\delta_t(x,y) = u_t(k_{t-1},k_t) + p_t k_t - p_{t-1}k_{t-1} - (u_t(x,y) + p_t y - p_{t-1}x)$, for any $(x,y) \in D_t$. By (4.12) $\delta_t(x,y) \geqq 0$.

(W3) *For any* $\varepsilon > 0$, *there is* $\delta > 0$, *such that* $|p_t(x-k_t)| > \varepsilon$ *implies* $\delta_{t+1}(x,y) > \delta$ *for any* $(x,y) \in D_{t+1}$.

Assumptions similar to these were used by Weizsäcker [1965] to prove existence for a one sector model that is time dependent.
(W2) places weak bounds on the limiting values of the capital stocks as $t \to \infty$, along feasible paths and along the path given by (W1). (W3) provides for a value loss for the input-output combination in period t when the value of input differs from the value of input on the given path. It is implied by uniform strict concavity of u along $\{k_t\}$, but it is weaker than that condition.

With these assumptions we may prove that $\{k_t\}$ is an optimal path. Consider any path $\{k'_t\}$ with $k'_0 = k_0$. Let $\delta_t = \delta_t(k'_{t-1},k'_t)$, $u_t = u_t(k_{t-1},k_t)$, $u'_t = u_t(k'_{t-1},k'_t)$. Then $u'_t - u_t = p_t(k_t-k'_t) - p_{t-1}(k_{t-1}-k'_{t-1}) - \delta_t$. Summing, we obtain

$$(1) \quad \sum_1^T (u'_t-u_t) = p_0(k'_0-k_0) + p_T(k_T-k'_T) - \sum_1^T \delta_t .$$

Since $k'_0 = k_0$, using (W2) gives

$$(2) \quad \limsup \sum_1^T (u'_t-u_t) \leqq M - M' - \lim \sum_1^T \delta_t .$$

Either $\{k_t\}$ catches up to $\{k'_t\}$ or $\limsup \sum_1^T (u'_t-u_t) > 0$. In the latter case (2) implies $\delta_t \to 0$. Then (W3) implies $p_T(k_T-k'_T) \to 0$, and (1) implies $\sum_1^T (u'_t-u_t) \leqq 0$ for large T, with < unless $k'_t \equiv k_t$. This means $\{k_t\}$ catches up to $\{k'_t\}$. Since $\{k'_t\}$ is an arbitrary path with $k'_0 = k_0$, $\{k_t\}$ catches up to every path from k_0 and $\{k_t\}$ is optimal. We have proved (McKenzie [1974])

<u>Theorem 3</u>. *Under Assumptions* (W1), (W2), *and* (W3), *the path* $\{k_t\}$ *is optimal.*

Once an optimal path $\{k_t\}$ has been shown to exist from the initial stock k_0, optimal paths may be derived from all initial stocks in the set K_0, that is, the set of stocks for which the value function is well defined after normalization of utility by $u_t(k_{t-1},k_t) = 0$, all t. The value function is well defined from a capital stock x if there exists a path $\{k'_t\}$ with $k'_0 = x$ such that $\liminf \sum_1^T u_t(k'_{t-1},k'_t) > -\infty$, as $T \to \infty$. Consideration of (2) shows that this condition is met if and only if the value loss $\sum_1^T \delta_t$ is bounded as $T \to \infty$. The value loss is the shortfall of the utility sum less a part due to the first differential of u, when u is differentiable, or an analog defined by the support function in the general concave case. The value loss method works because the first order effects on the utility sums depend only on the differences in value of the initial and terminal stocks, as (1) shows, and (W2) places certain bounds on the limiting values of the terminal stocks.

Let K be the set of capital stocks x with well defined values $V_0(x)$ when utility is normalized on the optimal path $\{k_t\}$. We prove (McKenzie [1974])

<u>Theorem 4</u>. *If there is an optimal path* $\{k_t\}$ *from* k_0, *satisfying Assumptions* I, II, *and* III, *and if* (W2) *and* (W3) *are satisfied for one of its supporting price sequences* $\{p_t\}$, *there is an optimal path from every capital stock in the set* K_0, *defined relative to* $\{k_t\}$.

By Lemma 1 the hypothesis of Theorem 4 implies (W1). Also from the discussion above, the set K_0 may equally well be defined as the set of stocks from which there exist paths with finite value loss. Let $L_0(x) = \inf(\lim \sum_1^T \delta_t(k'_{t-1},k'_t),\ T \to \infty)$ over paths $\{k'_t\}$ such that $k'_0 = x$. $L_0(x)$ is well defined if and only if $V_0(x)$ is well defined, given (W2) and (W3). However, $\sum_1^T \delta_t$ has the advantage over $\sum_1^T u_t$ that its terms are positive, so the finite sums converge if they are bounded above. This fact underlies the original Ramsey [1928] arguments for one sector models and was adapted to the multi-sector case by Atsumi [1965]. However, its full implications for the existence problem were first drawn by Brock [1970].

The essential step in proving Theorem 4 is to show that the infimum in the definition of $L_0(x)$ is assumed by a well defined path from x, if $x \in K_0$. This path will also realize the supremum in the definition of $V_0(x)$. Let s index a sequence of paths from x and let $L_0^s(x)$ be the value loss on the *sth* path. We may assume that the sequence is chosen so that $L_0^s(x) \to L_0(x)$. Let $\{k_t^s\}$ be the *sth* path. By Assumption II, k_t^s, s = 1,2,..., is bounded for each t. Thus we may use the Cantor diagonal process to choose a subsequence such that (retain

notation) $k_t^s \to \bar{k}_t$ for each t. By Assumption I, $(\bar{k}_{t-1},\bar{k}_t) \in D_t$. Otherwise $u(k^s_{t-1},k^s_t) \to -\infty$ and therefore, using (W2) in (1), $L^s(x) \to \infty$ and $x \notin K$. Then $\{\bar{k}_t\}$ is a path of accumulation from x.

Let $\bar{L}_0$ be the value loss associated with $\{\bar{k}_t\}$. Then $\bar{L}_0 \geq L_0(x)$. Suppose $\bar{L}_0 > L_0(x)$. Then for all large s, $\bar{L}_0 - L_0^s > \varepsilon$ for some $\varepsilon > 0$. Choose T so large that

(3) $\bar{L}_0 - \sum_1^T \delta_t(\bar{k}_{t-1},\bar{k}_t) < \varepsilon/4.$

Choose S so large that

(4) $\sum_1^T \delta_t(\bar{k}_{t-1},\bar{k}_t) - \sum_1^T \delta_t(k^s_{t-1},k^s_t) < \varepsilon/4,$ for s > S.

Then, adding (3) and (4), we have

(5) $\bar{L}_0 - \sum_1^T \delta_t(k^s_{t-1},k^s_t) < \varepsilon/2,$ for s > S.

But $L_0^s \geq \sum_1^T \delta(k^s_{t-1},k^s_t)$, so $\bar{L}_0 - L_0^s < \varepsilon/2$ for s > S which contradicts $\bar{L}_0 - L_0^s > \varepsilon$ for all large s. Therefore, $\bar{L}_0 = L_0(x)$, or the limit path realizes the minimal value loss over all infinite paths from x.

To prove that $\{\bar{k}_t\}$ is optimal we must show that it catches up to every path from x. Suppose there is a path $\{k'_t\}$ to which $\{\bar{k}_t\}$ does not catch up. Then lim sup $\sum_1^T (u'_t - \bar{u}_t) > 0$, as $T \to \infty$, where $u'_t = u_t(k'_{t-1},k'_t)$, $\bar{u}_t = u_t(\bar{k}_{t-1},\bar{k}_t)$. By normalization we may put $u_t(k_{t-1},k_t) = 0$, all t. Also $L_0(x)$ finite implies by (W3) that $p_T(k_T - \bar{k}_T) \to 0$, so by (1) $\sum_1^T \bar{u}_t \to p_0(x-k_0) - L_0(x) > -\infty$. If

$\sum_1^T \delta_t(k'_{t-1},k'_t) \to \infty$ with T, (2) implies $\lim\sup \sum_1^T u'_t \to -\infty$ as $T \to \infty$. Then $\lim\sup \sum_1^T (u'_t - \bar{u}_t) = \lim\sup \sum_1^T u'_t - \lim \sum_1^T \bar{u}_t = -\infty$ and $\{\bar{k}_t\}$ overtakes $\{k'_t\}$. On the other hand, if $\sum_1^T \delta_t(k'_{t-1},k'_t)$ is bounded as $T \to \infty$, (W3) again implies $p_T(k_T - k'_T) \to 0$, so by (1) $\sum_1^T u'_t \to p_0(x-k_0) - \sum_1^\infty \delta_t(k'_{t-1},k'_t)$. Thus we find

$$(6) \quad \sum_1^T (u'_t - \bar{u}_t) \to L_0(x) - \sum_1^\infty \delta_t(k'_{t-1},k'_t) \leqq 0, \text{ as } T \to \infty,$$

and $\{\bar{k}_t\}$ catches up to $\{k'_t\}$. Since $\{k'_t\}$ is an arbitrary infinite path from x, $\{\bar{k}_t\}$ is optimal, and Theorem 4 is proved.

We observe that $L_0(x)$ is finite and $V_0(x)$ is well defined relative to the optimal path $\{k_t\}$ if there exists a path $\{k''_t\}$ with $k''_0 = x$ and $k''_\tau = k_\tau$ for some $\tau \geqq 0$. Then $\{k_t\}$ is said to be *reachable* from x. In stationary models this is often provided for relative to the stationary optimal path.

It is clear from (1) that $\lim\sup \sum_1^T (u'_t - u_t) \leqq \lim\sup \sum_1^T p_T(k_T - k'_T)$. Thus $\{k_t\}$ is optimal if $\lim\sup \sum_1^T p_T(k_T - k'_T) \leqq 0$ holds for all paths $\{k'_t\}$ with $k'_0 = k_0$. In particular, if $\lim p_t = 0$ and k_t is bounded over t, $\{k_t\}$ is optimal. These conditions are likely to be met in models where $u_t = \rho^t u$ for $0 < \rho < 1$, which will be called quasi-stationary. We may state

(W2') k_t *is bounded over* t *and* $\lim p_t = 0$.

Then we have

Theorem 5. *Under Assumptions* I, II, (W1), *and* (W2') *the path* $\{k_t\}$ *is optimal.*

Assumption (W2') was introduced in the efficiency context by Malinvaud [1953] in the form $p_t k_t \to 0$, as $t \to \infty$.

6. The Stationary Model

A particular model to which Theorems 3 and 4 may be applied is that of Gale [1967] and McKenzie [1968]. In this model the utility function is stationary, that is, $D_t = D$ and $u_t = u$ for all t. It may be shown that a constant path that gives maximum sustainable utility (that is, $k_t = k$, all t, and $u(k,k) \geqq u(x,y)$ for $(x,y) \in D$ and $y \geqq x$) is supported by prices in the sense of (4.12), so that it satisfies (W1). Since the prices may also be chosen to be constant and any path is bounded in this model, (W2) follows directly. (W3) also follows if u(x,y) is strictly concave at (k,k). Then Theorems 1 and 2 apply to show that $\{k_t\}$, t = 0,1,..., where $k_t = k$, all t, is optimal, and there is an optimal path from every $x \in K_t = K$, where K_t is defined relative to $\{k_t\}$ as in Section 4. On the assumptions often adopted in the stationary model K includes all positive stocks and all stocks from which positive stocks may be reached. Free disposal of surplus stocks, the expansibility of certain stocks, and $0 \in D$ are used to imply the wide scope of K.

In order to have a set of assumptions that imply (W1), (W2), and (W3) and are somewhat more specific than those conditions, we will describe the stationary model. The assumptions I and II of the basic model are retained and in addition we assume

(G1) $D_t = D$, $u_t = u$, *for all* t. (Stationarity)

(G2) *There is* $\zeta > 0$ *such that* $|x| > \zeta$ *implies for any* $(x,y) \in D$ *that* $|y| < \gamma|x|$ *for* $\gamma < 1$. (Bounded paths)

(G3) *If* $(x,y) \in D$, *then* $(z,w) \in D$ *for all* $z \geqq x$, $0 \leqq w \leqq y$, *and* $u(z,w) \geqq u(x,y)$ holds. (Free disposal)

(G4) *There is* $(\bar{x},\bar{y}) \in D$ *for which* $\bar{y} > \bar{x}$. (Existence of an expansible stock)

Before stating the last assumption we must show that a constant path exists with constant prices satisfying (W1). Define the set $V = \{v \mid v = y-x$, where $(x,y) \in D\}$. Since $E_t = E^n$, an n dimensional Euclidean space, all t, $V \subset E^n$. By free disposal, (G3), and, (G4), the existence of an expansible stock $\bar{x}$, $0 \in$ interior V. Indeed, $(x',y') \in V$ if $x < x' < y$ and $x < y' < y$. We will show that $v = y-x \geqq 0$ implies (x,y) is bounded. By (G4) there is $\bar{v} = \bar{y}-\bar{x} > 0$. Suppose there is $v < 0$ such that $D_v = \{(x,y) \mid v = y-x\}$ is not bounded. Choose α to give $v' = \alpha\bar{v} + (1-\alpha)v = 0$, where $0 < \alpha < 1$. Let $(x',y') = \alpha(\bar{x},\bar{y}) + (1-\alpha)(x,y)$ for $(x,y) \in D_v$. Then $v' = y'-x' = 0$ but (x',y') can be chosen arbitrarily large by choosing $(x,y) \in D_v$ arbitrarily large, contradicting (G2). Thus D_v is bounded for any $v \in V$.

Define $f(v) = \sup u(x,y)$ for $(x,y) \in D_v$. Since u is concave and closed by Assumption I and D_v is bounded, the sup is attained for any $v \in V$. Let $W = \{(u,v) \mid u \leq f(v)$ and $v \in V\}$. W is convex since f is concave and, putting $\bar{u} = f(0)$, $(\bar{u},0)$ is a boundary point of W. Thus there is $(\pi,p) \in E^{n+1}$ and $(\pi,p) \neq 0$, such that $\pi u + pv \leq \pi\bar{u}$ for all $(u,v) \in W$. Since V is unbounded below by (G3), $p \geqq 0$. Suppose $\pi = 0$. Then $pv \leq 0$ for all $v \in V$, or since 0 is interior to V, $p = 0$. Thus $\pi \neq 0$, and we may choose (π,p) so that $\pi = 1$. Then $u + pv \leq \bar{u}$ for all $(u,v) \in W$. This implies

<u>Lemma 2</u>. *There is* $p \geqq 0$ *such that* $u(x,y) + py - px \leq \bar{u}$ *for all* $(x,y) \in D$, *where* $\bar{u} = \max u(x,x)$ *for* $(x,x) \in D$.

Let $u(k,k) = \bar{u}$. Then the path $\{k_t\}$, $t = 0,1,\ldots$, with $k_t = k$ for all t is an infinite path supported, in the sense of (4.12), by the price sequence $\{p_t\}$, where $p_t = p$ for all t. We now assume

(G5) *The utility function* u *is strictly concave near the point* (k,k) *where* $u(k,k) \geq u(x,x)$ *for all* $(x,x) \in D$.

It follows from (G5) that $u(x,x) = \bar{u}$ implies $(x,x) = (k,k)$.

The value loss relative to the constant path $k_t = k$ is $\delta(x,y) = \bar{u} - u(x,y) - py + px$. Then $\delta(x,y) \geqq 0$. We may use (G5) to prove (Atsumi [1965], Radner [1961]).

<u>Lemma 3</u>. *For any* $\varepsilon > 0$, *there is* $\delta > 0$, *such that* $|p(x-k)| > \varepsilon$ *implies* $\delta(x,y) > \delta$ *for any* $(x,y) \in D$. *Here* $u(k,k) = \bar{u}$ *and* p *and* $\bar{u}$ *are given by Lemma* 2.

Suppose Lemma 3 is not true. Then there exists a sequence (x^s,y^s), $s = 1,2,\ldots$, such that $|x^s-k| > \varepsilon$ and $\delta(x^s,y^s) \to 0$. Since by concavity $\delta(x^s,y^s)$ is falling as (x^s,y^s) approaches (k,k) along a line segment, we may put $|x^s-k| = \varepsilon$ for all s. Then the sequence (x^s,y^s), which is bounded by II, has a point of accumulation $(\bar{x},\bar{y})$ where $\delta(\bar{x},\bar{y}) = 0$ and $(\bar{x},\bar{y}) \in D$ by concavity and closedness of u on D. Then strict concavity implies $\delta(x,y) < 0$ for (x,y) between $(\bar{x},\bar{y})$ and (k,k) in contradiction to Lemma 2.

Lemma 3 implies (W3) for $k_t = k$, $p_t = p$. Lemma 2 implies (W1), and (W2) follows directly from $k_t = k$ and $p_t = p$. Thus Theorem 1 implies that $k_t = k$, $t = 0,1,\ldots$, is an optimal path, and from (G5) it is unique. Also Theorem 2 implies that an optimal path exists from any $x \in K$, that is, from any x for which $V(x) > -\infty$ or equivalently $L(x) < \infty$, where these functions are defined relative to the stationary optimal path.

On the basis of Lemma 3 we may show that the prices derived in Lemma 2 are full Weitzman prices, that is,

<u>Corollary</u>. *Given Assumption* (G5) *the prices* (p,p) *of Lemma* 2 *and* (k,k), *where* $u(k,k) = \bar{u}$, *satisfy both* (4.11) *and* (4.12). *Moreover,* $k_t = k$ *is the only stationary path with stationary price supports.*

That (p,p) and (k,k) satisfy (4.12) is the content of Lemma 2. Uniqueness follows from (G5) and the fact that (4.12) with $p_t = p_{t+1}$ implies $u(k,k) = \bar{u}$. Let k_0 lie in K and consider

(1) $u(k,k) = u(k_{t-1},k_t) + qk_t - qk_{t-1} + \delta_t(k_{t-1},k_t)$,

where $\{k_t\}$ is any path from k_0. Summing (1) gives

(2) $Tu = \sum_1^T u_t + qk_T - qk_0 + \sum_1^T \delta_t.$

Let $u(k,k) = 0$. Then $\liminf \sum_1^T u_t \leqq V_0(k_0) > -\infty$ as $T \to \infty$. Also by Lemma 3 $k_T \to k$, or $\sum \delta_t \to \infty$. Therefore, taking lim inf of (2) we obtain

(3) $V_0(k) - qk \geqq V_0(k_0) - qk_0,$

where $V_0(k) = 0$. However, (3) is (4.11) for the present case.

We say that a stock x is *expansible* if there is $(x,y) \in D$ with $y > x$. We can prove (Gale [1967])

Lemma 4. *If* x *is expansible, then* $x \in K$.

Consider $\alpha^t(x,y) + (1-\alpha^t)(k,k) = (k_t,k'_{t+1})$, where $y > x$, $0 < \alpha < 1$, $t = 0,1,2,\ldots$. For $t = 0$, $(k_t,k'_{t+1}) = (x,y)$, and as $t \to \infty$, $(k_t,k'_{t+1}) \to (k,k)$. But $k'_{t+1} = k - \alpha^t(k-y)$ and $k_{t+1} = k - \alpha^{t+1}(k-x)$. Then $k'_{t+1} > k_{t+1}$ if $y-\alpha x > (k-\alpha k)$. This holds for α near 1 since $y > x$. Thus by free disposal $\{k_t\}$ is an infinite path approaching k. The value loss function δ is convex since u is concave. Also

$$(k_t,k_{t+1}) = (k,k) - \alpha^t((k,k) - (k_0,k_1)).$$

Therefore,

$$\delta_t(k_{t-1},k_t) \leqq (1-\alpha^t)\delta(k,k) + \alpha^t\delta(k_0,k_1) = \alpha^t\delta(k_0,k_1),$$

and $\sum_1^\infty \delta_t(k_{t-1},k_t) \leqq \frac{1}{1-\alpha}\,\delta(k_0,k_1)$, proving that $k_0 = x \in K$.

Summarizing the above results we may state

Theorem 6. *If in addition to Assumptions* I *and* II *we accept Assumptions* (G1) - (G5), *there is a unique optimal stationary path supported by stationary price vectors, in the sense of* (4.11) *and* (4.12), *and there is an optimal path from any expansible stock.*

Without the assumption of strict concavity at the stationary path that maximizes stationary utility, we cannot show that expansible stocks give rise to optimal paths. However, on the weaker assumption that this path is unique the analogous result can be proved for maximal paths. See Brock [1970], where the terminology "weakly maximal" is used. The appropriate assumption is

(G5') *There is a point* $(k,k) \in D$ *such that* $u(y,y) \geq u(x,x)$ *for all* $(x,x) \in D$ *implies* $(y,y) = (k,k)$.

This assumption is only slightly weaker than requiring u to be strictly concave at (k,k) in the direction of the diagonal. But the possibility that u has a flat contour in other directions means that other paths originating at k may exist which oscillate about k without suffering value losses. See McKenzie [1968].

Make Assumptions (G1) - (G4) and (G5'). Let p be the price vector of Lemma 2, where $u(k,k) = \bar{u}$. As before, define the set K relative to the path $k_t = k$, $t = 0,1,\ldots$, where $K = \{x|V_0(x) > -\infty\}$. Equivalently $K = \{x|L_0(x) < \infty\}$, where $L_0(x)$ is defined relative to $k_t = k$ and $p_t = p$. As in the proof of Theorem 4, for $x \in K$ there is a path $\{k'_t\}$, $t = 0,1,\ldots$, that realizes minimum value loss $L_0(x)$ when $k'_0 = x$. It is implied by (G2) that $\{k'_t\}$ is bounded. Thus $\frac{1}{T}\sum(k'_{t-1},k'_t) = (\bar{k}_{T-1},\bar{k}_T)$ has a limit point $(\bar{k},\bar{k})$. By closedness of u, $(\bar{k},\bar{k}) \in D$. Let $u(k,k) = 0$. Then

(4) $\sum_{1}^{T} u(k'_{t-1},k'_t) = p(k'_0-k'_T) - \sum_{1}^{T}\delta(k'_{t-1},k'_t)$

from (5.1). Since $\sum_{1}^{T}\delta'_t \to L_0(x)$, (4) implies $\frac{1}{T}\sum_{1}^{T}u'_t \to 0$. On the other hand, by by concavity of u, $\frac{1}{T}\sum_{1}^{T}u'_t \leqq u(\bar{k}_{T-1},\bar{k}_T)$. Thus $u(\bar{k},\bar{k}) = 0$ and $\bar{k} = k$ by Assumption (G5').

Suppose $\{k''_t\}$ is any other path from x. As in the proof of Theorem 4 it is enough to consider paths with finite value loss. Then by the above argument $\frac{1}{T}\sum k''_t$ also converges to k. However, from (5.1) we derive

(5) $\sum_{1}^{T}(u(k''_{t-1},k''_t) - u(k'_{t-1},k'_t)) = p\cdot(y'_T-y''_T) + \sum_{1}^{T}\delta_t(k'_{t-1},k'_t) - \sum_{1}^{T}\delta_t(k''_{t-1},k''_t).$

Suppose $\liminf \sum_{1}^{T}(u''_t-u'_t) = \gamma > 0$. Since $\sum_{1}^{T}\delta'_t$ is minimal, $\lim(\sum_{1}^{T}\delta'_t-\sum_{1}^{T}\delta''_t) \leqq 0$. Thus (5) implies $\liminf p\cdot(y'_T-y''_T) \geqq \gamma$ must hold. But $y'_T \to k$ and $y''_T \to k$, which is a contradiction. Thus $\liminf \sum_{1}^{T}(u''_t-u'_t) \leqq 0$ and $\{k'_t\}$ is maximal. This establishes

Theorem 7. *If in addition to* I *and* II *we accept Assumptions* (G1) - (G4) *and* (G5'), *there is a maximal path from any expansible stock.*

7. The Quasi-Stationary Model

The quasi-stationary model differs from the stationary model by the presence of a discount factor $0 < \rho < 1$ for utility, that is, $u_t(x,y) = \rho^t u(x,y)$ for $t \geq 0$. We will first prove that a quasi-stationary model has an optimal stationary path that is supported by proportional price vectors (Sutherland [1970]). As for the stationary model, from this path we may derive the existence of other optimal paths.

For the quasi-stationary model, we assume, in addition to I and II of the basic model,

(S1) $D_t = D \subset E^{2n}$, $u_t = \rho^t u$, *for all* t, *where* $0 < \rho < 1$. (Quasi-stationarity)

(S2) Identical with (G2). (Bounded paths)

(S3) Identical with (G3). (Free disposal)

(S4) *There is* $(\bar{x},\bar{y}) \in D$ *for which* $\rho\bar{y} > \bar{x}$. (Existence of a stock expansible by the factor ρ^{-1})

These assumptions are small modifications of those for the stationary model of section 6, (G1) - (G4), to allow for the presence of ρ. Indeed, if ρ is put equal to 1, they are the same.

We will show that an optimal stationary oath exists in the quasi-stationary model. This extends a result due to Peleg and Ryder [1972] to a general reduced form model. Let Δ be the set $\{(x,x)|x \geqq 0$ and $|x| \leqq \zeta\}$. Δ is a compact convex subset of the diagonal of $E^n \times E^n$. For any $(x,x) \in \Delta$ define $f(x,x) = \{(z,w)|\rho w-z \geqq (\rho-1)x$ for $(z,w) \in D\}$. Since it contains the point $(\bar{x},\bar{y})$ by Assumption (S4), f(x,x) is not empty. Moreover, $(z,w) \in f(x,x)$ implies $|z| \leqq \zeta$. To see this consider

(1) $w \geqq \alpha(z-x) + x,$

for $\alpha = \rho^{-1} \geqq 1$. If equality holds in (1) and $\alpha = 1$, $|w| = |z|$, and $\frac{\partial |w|}{\partial \alpha} > 0$ for $\alpha \geq 1$ and $|z| > |x|$. However, $|z| > \zeta$ implies $|z| > |x|$ and thus $|w| > |z|$. This contradicts Assumption (S2), so in this case $|z| \leq \zeta$. But if inequality holds in (1), the argument is true *a fortiori*. Also (S2), together with free disposal (S3), implies that $|w| < \zeta$ if $|z| \leq \zeta$. Thus the set $f(x,x)$ is bounded.

For $U \subset D$, let $g(U) = \{(z,w) \in U | u(z,w) \geqq u(z',w') \text{ for all } (z',w') \in U\}$. Consider $U = f(x,x)$. Since $u(x,y)$ is concave and closed by Assumption 1 and $f(x,x)$ is bounded, the set $\{(z,w) \in U | u(z,w) \geqq u(x,x)\}$ is compact. Therefore, $g(U)$ is compact and not empty. Also by concavity, $g(U)$ is convex. Let $h(W)$, for $W \subset D$, be the set $\{(z,z) | (z,w) \in W\}$, which lies in Δ. Thus h is a projection on Δ along the first factor of the Cartesian product $E^n \times E^n$. Finally, define the correspondence $F = h \circ g \circ f$. F maps Δ into the set of non-empty, convex, compact subsets of Δ. See Figure 2.

We will need

Lemma 5. *The correspondence* f *is lower semicontinuous.*

Suppose $(z,w) \in f(x,x)$ and $(x^s,x^s) \to (x,x)$, where $(x^s,x^s) \in \Delta$, $s = 1,2,\ldots$. Let (z^s,w^s) be the point on the line segment from the point $(\bar{x},\bar{y})$ assumed in (S4) to (z,w) that is closest to (z,w) and also contained in $f(x^s,x^s)$. The existence of such a point follows from $(\bar{x},\bar{y}) \in f(x^s,x^s)$. If $(z^s,w^s) \not\to (z,w)$, there must be an accumulation point $(\bar{z},\bar{w})$ separated from (z,w) by a distance of at least ε and a subsequence (z^s,w^s)(retain notation) such that $(z^s,w^s) \to (\bar{z},\bar{w})$ and $\rho w^s - z^s = (\rho-1)x^s$. Since $(\rho\bar{y}-\bar{x}) > (\rho-1)x$ and $(\rho w-z) \geqq (\rho-1)x$ it follows that $(\rho\bar{w}-\bar{z}) > (\rho-1)x$. This implies that $(\rho w^s - z^s) > (\rho-1)x^s$ for large s, contradicting the choice of (z^s,w^s). Thus $(z^s,w^s) \to (z,w)$ and f is lower semi-continuous.

The continuity of u and the lower semi-continuity of f imply that $g \circ f$ is upper semi-continuous (Berge [1963], p. 116). Since h is a continuous

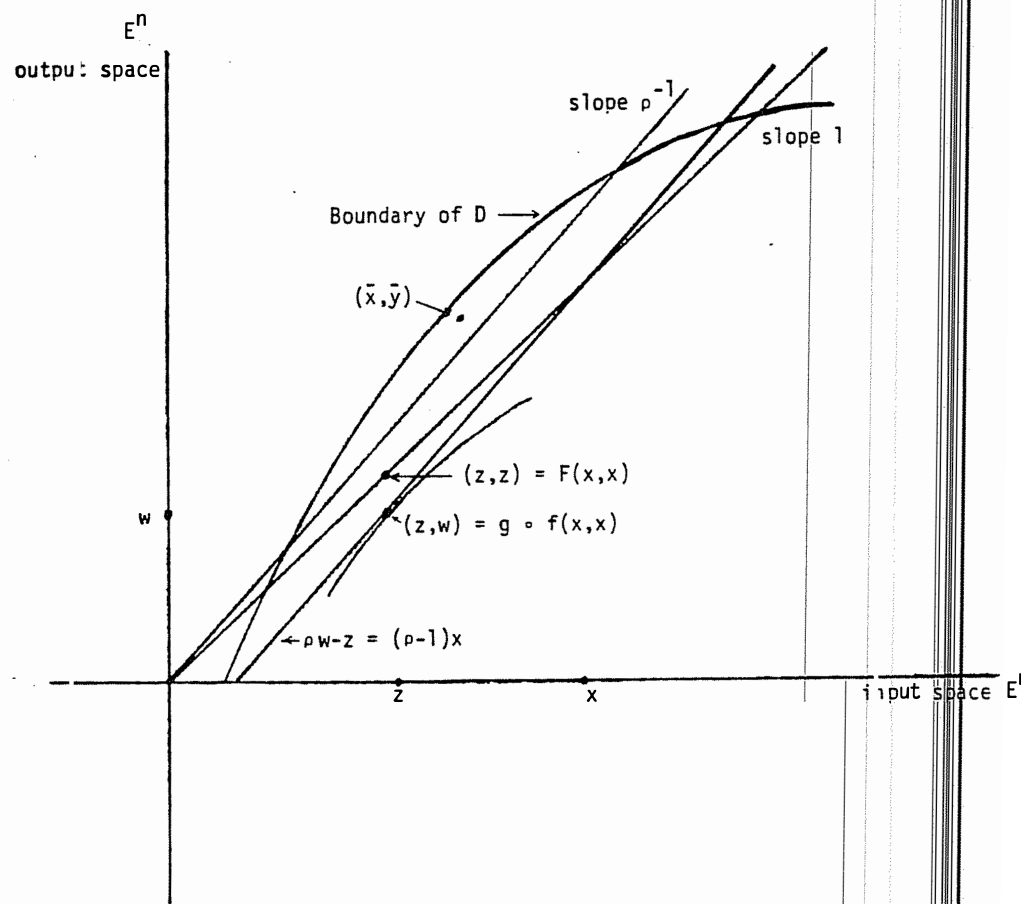

Figure 2

correspondence, $F = h \circ g \circ f$ is upper semi-continuous. Since Δ is compact and convex and F maps Δ into convex subsets, the Kakutani fixed point theorem (Berge [1963], p. 174) implies there is (k,k) such that $(k,k) \in F(k,k)$. We will show that (k,k) is a stationary path supported by proportional price vectors. It is clear that (k,k) maximizes utility over $f(k,k)$. If not, there is $(k,w) \neq (k,k)$ that does and, by definition of f, $w \geq k$. Then by free disposal (S3), (k,k) also maximizes utility over $f(k,k)$.

The derivation of price supports for (k,k) parallels that of section 6. Define the set $V = \{v | v = \rho w - z, \text{ for some } (z,w) \in D\}$. By free disposal, (S3), and the existence of an expansible stock, (S4), $(\rho-1)k \in$ interior V. For $v \in V$, let $D_v = \{(z,w) \in D | \rho w - z \geq v\}$. D_v is bounded for any $v \in V$ by an argument parallel to that given in Section 6 for $\rho = 1$. Define $\phi(v) = \sup u(x,y)$ for $v = \rho y - x$, $v \in V$. The sup is attained as before. Let $W = \{(u,v) | u \leq \phi(v), v \in V\}$. W is convex. Let $\bar{v} = (\rho-1)k$ and $\bar{u} = \phi(\bar{v})$. Then $(\bar{u},\bar{v})$ is a boundary point of W. Thus there is $(\pi,q) \in E_{n+1}$ and $(\pi,q) \neq 0$, such that $\pi u + qv \leq \pi\bar{u} + (\rho-1)qk$ for all $(u,v) \in W$. Since v is unbounded below by (S3), $q \geq 0$ must hold. Suppose $\pi = 0$. Then $qv \leq (\rho-1)qk$ for all $v \in V$. However, $(\rho-1)k$ is interior to V, so $q = 0$. Thus $\pi \neq 0$, and we may choose (π,q) so that $\pi = 1$. Then $u + qv \leq \bar{u} + (\rho-1)qk$ for all $(u,v) \in W$. This implies

Lemma 6. *There is* $(k,k) \in D$ *and* $q \geq 0$ *such that* $u(z,w) + \rho qw - qz \leq \bar{u} + \rho qk - qk$ *for all* $(z,w) \in D$, *where* $\bar{u} = u(k,k)$.

Consider the path $\{k_t\}$, $t = 0,1,\ldots$, where $k_t = k$, all t, and the vectors q and k satisfy Lemma 6. Then the price path $\{p_t\}$, $t = 0,1,\ldots$, where $p_t = \rho^t q$ supports the utility function u in the sense of (4.12). It is clear that $p_t \to 0$ and k_t is bounded over t, or (W2') holds. Thus by Theorem 5, the path $\{k_t\}$ is optimal. An examination of the proof of Theorem 4 shows that (W2') will also replace (W2) there. Then given (W2'), it is unnecessary to use (W3) to show that $p_T(k_T - k_T') \to 0$, and (5.6) is established directly. Thus Theorem 5

is valid with (W2') replacing both (W2) and (W3), and an optimal path exists from any $x \in K$, that is, from any x for which $V_0(x) > -\infty$, where this function is defined relative to the stationary optimal path.

We may also show that (4.11) holds for p_t so that they are full Weitzman prices. Consider

$$(2) \quad \rho^t u(k,k) + (\rho^t - \rho^{t-1})qk = \rho^t u(k_{t-1},k_t) + \rho^t qk_t - \rho^{t-1} qk_{t-1} + \delta_t(k_{t-1},k_t),$$

where $\{k_t\}$ is a path from k_0 and $\delta_t(k_{t-1},k_t) \geqq 0$. Summing (2) gives

$$(3) \quad \sum_1^T \rho^t u + (\rho^T - 1)qk = \sum_1^T \rho^t u_t + \rho^T qk_T - qk_0 + \sum_1^T \delta_t.$$

In the limit (3) justifies

$$V_0(k) - qk \geqq V_0(k_0) - qk_0,$$

which establishes (4.11).

In this case it is not difficult to show that the set K of capital stocks x, with well defined values $V_0(x)$ relative to the stationary optimal path $k_t = k$, includes all sustainable stocks. If x is sustainable, that is, $(x,x) \in D$, then one feasible path from x is $k_t = x$, $t = 0,1,\ldots$. This implies $V_0(x) \geqq \sum \rho^t u(x,x) = \frac{1}{1-\rho} u(x,x)$, so $V_0(x) > -\infty$ holds. Thus we have

Theorem 8. *If in addition to Assumptions* I *and* II *we accept Assumptions* (S1) - (S4), *there is a stationary optimal path* $k_t = k$, *supported by price vectors* $p_t = \rho^t q$ *in the sense of* (4.11) *and* (4.12). *Also there is an optimal path from any sustainable stock.*

According to Lemma 6, under the conditions assumed, there always exists an optimal stationary path $k_t = k$ supported by a price sequence $p_t = \rho^t q$, that is, by proportional prices. We may also show that any stationary optimal path (k,k) has proportional supports if $(k,k) \in$ interior D. Since $k_t = k$, $t = 0,1,\ldots$, is an optimal path interior to D it satisfies the hypothesis of Lemma 1. Thus a sequence of support prices $\{p_t\}$, $t = 0,1,\ldots$, exists, and $p_t \geqq 0$ by free disposal. Consider prices (p,q) that support $u(k,k)$, that is,

$$(4)\quad u(x,y) - u(k,k) \leqq p(x-k) - q(y-k), \text{ for all } (x,y) \in D.$$

Since $(k,k) \in$ interior D, it is immediate that there is β such that $|p| < \beta$ and $|q| < \beta$.

By the support property we have

$$\rho^t u(x,y) + p_t y - p_{t-1} x \leqq \rho^t u(k,k) + p_t k - p_{t-1} k.$$

Dividing through by ρ^t gives

$$(5)\quad u(x,y) - u(k,k) \leqq \frac{p_t}{\rho^t}(x-k) - \frac{p_{t-1}}{\rho^t}(y-k),\ t = 1,2,\ldots.$$

averaging the first T+1 inequalities (5), gives

$$(6)\quad u(x,y) - u(k,k) \leqq P_T(x-k) - Q_T(y-k),$$

where $P_T = \frac{1}{T+1}(p_0 + \rho^{-1}p_1 + \ldots + \rho^{-T}p_T)$, and

$$Q_T = \frac{1}{T+1}(p_1 + \rho^{-1}p_2 + \ldots + \rho^{-T}p_{T+1})$$

$$= \rho P_T + \frac{\rho}{T+1}(\rho^{-T-1}p_{T+1} - p_0).$$

Since $|P_T| < \beta$ for all T, there is a subsequence $\{T_i\}$, $i = 1,2,\ldots$, such that $P_{T_i} \to q \geqq 0$. Also Q_{T_i} converges to ρq, and we obtain from (6)

$$(7)\quad u(x,y) - u(k,k) \leqq q(x-k) - \rho q(y-k).$$

Then the price sequence $\{p_t'\}$, where $p_t' = \rho^t q$, gives proportional support prices for $k_t = k$. This argument is due to Sutherland [1970]. We shown

Lemma 7. *The path* $\{k_t\}$, $k_t = k$, $t = 0,1,\ldots$, *where* (k *interior* D, *is an optimal path given* (S1) - (S4), *if and only if there are support prices* $\{p_t\}$ *where* $p_t = \rho^t q$, $q \geqq 0$, *that satisfy* (4.12).

It is implied by Lemma 7 that $u(k,k)$ maximizes $u(x,y)$ subject to $\rho y - x = (\rho-1)k = v$. Consider $y = \rho^{-1}(x+v)$, $k = \rho^{-1}(k+v)$. Substituting in (7), we have

$$u(x,y) - u(k,k) \leqq q(x-k) - \rho q(\rho^{-1}x - \rho^{-1}k)$$

or $u(x,y) - u(k,k) \leqq 0$. However, it is clear from Theorem 8 and the proof of Lemma 6 that if (k,k) maximizes $u(x,y)$ subject to $\rho y - x = (\rho-1)k$, $k_t = k$ is optimal. Then we have the

Corollary. *The path* $\{k_t\}$, $k_t = k$, $t = 0,1,\ldots$, *where* $(k,k) \in$ *interior* D, *is an optimal path given* (S1) - (S4) *if and only if* $u(k,k)$ *maximizes* $u(x,y)$ *for* $\rho y - x = (\rho-1)k$.

The necessity parts of Lemma 7 and the Corollary also apply to the stationary model, since the same arguments are valid. However for sufficiency Assumption (G5) would be needed, that u is strictly concave at (k,k).

8. Convergence of Optimal Paths

There are three general methods available for proving the convergence of optimal paths. A very simple method may be used when the utility function is uniformly concave, in a certain sense, along an optimal path. This method makes direct use of the fact that a chord of the graph of the utility function lies entirely below the graph. However, the method is too weak for cases where uniform concavity does not hold. Then a dual approach is needed based upon the support prices. This approach has been referred to as the method of "value loss," since it is the accumulation of shortfalls in values of input-output combinations along one path relative to another at the other's support prices that eventually contradicts optimality. However, it is not first order value losses that force convergence. They are fully accounted for over a segment of the optimal paths by the differences in value of initial and terminal stocks. Rather the work is done by second order value losses due to concavity. Thus our arguments are closely related to the problem of the second variation in calculus of variations. This analogy may be illuminating to students of the calculus. However, it should be kept in mind that turnpike theory compares paths starting from different points both of which are maximal relative to their starting points. This is unlike the classical problems of calculus of variations. Finally a method is available based on the treatment of the first order conditions for optimality as a set of difference equations that define a transformation of the paths of accumulation into a Banach space. This approach will be examined in Section 10.

Let $\{k_t\}$ and $\{k_t'\}$ be two optimal paths for $t = 0,1,\ldots$, where k_0 and k_0' may differ. Assume I, II, and $k_0 \in \text{int } K_0$, and suppose $k_0' \in K_0$, that is, $V_0(k_0') > -\infty$, when utility is normalized on $\{k_t\}$. The primal approach to convergence considers a path that is halfway between $\{k_t\}$ and $\{k_t'\}$, that is,

$\{k_t''\}$, where $k_t'' = \frac{1}{2}(k_t+k_t')$. By convexity, $k_t'' \in K_t$ for all t. Assume *uniform strict concavity* of u_t along $\{k_t\}$ in the primal sense that $|(x,y) - (k_{t-1},k_t)| > \varepsilon > 0$ implies there is $\delta > 0$, independent of t, such that

$$(1)\quad u_t(\tfrac{1}{2}(x+k_{t-1},y+k_t)) \geqq u_t(x,y) + u_t(k_{t-1},k_t) + \delta.$$

Applying (1) to $\{k_t\}$, $\{k_t'\}$ suppose the distance between the paths exceeds ε, $s(T)$ often by time T. Put $u_t(k_{t-1},k_t) = 0$ for all t. Then

$$(2)\quad \sum_1^T u_t(k_{t-1}'',k_t'') \geqq \sum_1^T u_t(k_{t-1}',k_t') + s(T)\delta.$$

If $s(T) \to \infty$ as $T \to \infty$, $\sum_1^T u_t'' \to \infty$ and $V_0(k_0'') = \infty$. Since $k_0 \in \text{int } K_0$, $V_0(k_0) = \infty$ would be implied as we saw in Section 4 in contradiction to $V_0(k_0) = 0$ by the normalization. More exactly we may prove (Jeanjean [1974])

<u>Theorem 9</u>. *Let* $\{k_t\}$, $\{k_t'\}$, $t = 0,1,\ldots$, *be optimal paths and assume* I *and* II, *and* $k_0 \in$ *relative int* K_0. *Assume uniform strict concavity of* u_t *along* $\{k_t\}$. *Suppose* $k_0' \in K_0$. *Then for any* $\varepsilon > 0$ *there is a number* $N(\varepsilon)$ *such that* $|k_t'-k_t| > \varepsilon$ *can hold for at most* $N(\varepsilon)$ *periods.*

To find $N(\varepsilon)$ let $\bar{k} \in K_0$, where $k_0 = \alpha\bar{k} + (1-\alpha)k_0''$ for some α, $0 < \alpha < 1$. Then, by concavity of V_0, $0 = V_0(k_0) \geqq \alpha V_0(\bar{k}) + (1-\alpha)V_0(k_0'')$, or $V_0(k_0'') \leqq \frac{\alpha}{\alpha-1} V_0(\bar{k})$. At the same time, $V_0(k_0'') \geqq \frac{1}{2}(V_0(k_0) + V_0(k_0')) + N(\varepsilon)\delta = \frac{1}{2}V_0(k_0') + N(\varepsilon)\delta$. Thus $N(\varepsilon) \leqq \delta^{-1}(\frac{\alpha}{\alpha-1}V_0(\bar{k}) - \frac{1}{2}V_0(k_0'))$, which may be seen to be positive. This proves the theorem.

In the stationary model of Section 5, where $u_t = u$, $K_t = K$, for all t, uniform strict concavity at the (k,k) of Assumption (G5) is immediate, and

Theorem 9 implies that all expansible stocks lead to optimal paths that converge to the stock k of the optimal stationary path. This result was first proved in a multi-sector model by Atsumi [1965], using the value loss approach.

If the hypothesis of Theorem 9 is strengthened by including the first part of Assumption III, so that support prices may be shown to exist, the dual approach may be used to draw the conclusion of the theorem (McKenzie [1976]). In this case it is convenient to use a dual notion of uniform value loss. The definition of *uniform value loss* along (k_t,p_t) is that $|(x,y) - (k_{t-1},k_t)| > \varepsilon > 0$ implies there is $\delta_{t+1}(x,y) > \delta$ for all t. Since this notion is weaker than the primal notion of uniform strict concavity, the two versions of Theorem 9 have no simple order of strength.

The role of uniform strict concavity in the value loss approach is to provide uniform value loss when $(x,y) \neq (k_t,k_{t+1})$. The value loss in period t+1 for capital stocks (x,y) relative to the path $\{k_\tau\}$, supported by prices $\{p_\tau\}$, was defined in Section 5 by

$$(3)\quad u_{t+1}(k_t,k_{t+1}) + p_{t+1}k_{t+1} - p_tk_t = u_{t+1}(x,y) + p_{t+1}y - p_tx + \delta_{t+1}(x,y).$$

From (4.12) the value loss $\delta_{t+1}(x,y)$ is well defined and non-negative for all $(x,y) \in D_{t+1}$. If strict concavity holds, it is also positive for $(x,y) \neq (k_t,k_{t+1})$. Indeed, by the same proof used for Lemma 3, we obtain

Lemma 8. *If* u_t *satisfies* I *and is strictly concave at* (k_t,k_{t+1}), *for any* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $|x-k_t| > \varepsilon$ *implies* $\delta_{t+1}(x,y) > \delta$, *for any* $(x,y) \in D_{t+1}$.

Let us consider two paths $\{k_t\}$ and $\{k'_t\}$, t = 0,1,..., that are maximal where k_0 and k'_0 may differ. Assume I, II, and III for $\{k_t\}$. Suppose $V_0(k'_0) > -\infty$ when utility is normalized on $\{k_t\}$, and $k'_0 \in \text{int } K_0$. As

we have seen, $V_0(k'_0) < +\infty$. Then $V'_0(k_0) = -V_0(k'_0) > -\infty$ where V'_0 is derived from normalizing utility on $\{k'_t\}$. It is clear that Assumption III holds relative to either path and $K_t = K'_t$ for all t. Thus support prices exist for both paths by Lemma 1. Also $V_t(k'_t) = -V'_t(k_t)$ is well defined and finite for all t.

The definition of the value losses in (3) gives symmetrical expressions for the two paths,

$$(4)\quad u_t(k_{t-1},k_t) + p_tk_t - p_{t-1}k_{t-1} = u_t(k'_{t-1},k'_t) + p_tk'_t - p_{t-1}k'_{t-1} + \delta_t,$$

$$(5)\quad u_t(k_{t-1},k_t) + p'_tk_t - p'_{t-1}k_{t-1} = u_t(k'_{t-1},k'_t) + p'_tk'_t - p'_{t-1}k'_{t-1} - \delta'_t.$$

In these formulae, $\delta_t = \delta_t(k'_{t-1},k'_t)$, and $\delta'_t = \delta'_t(k_{t-1},k_t)$. The prices and thus the size of value losses are independent of the normalization of u_t. Subtracting (5) from (4) gives

$$(6)\quad (p'_t-p_t)(k'_t-k_t) - (p'_{t-1}-p_{t-1})(k'_{t-1}-k_{t-1}) = \delta_t + \delta'_t.$$

Let $L_p(t) = (p'_t-p_t)(k'_t-k_t)$.

We may apply the support of the value function according to (4.11) to obtain

$$(7)\quad V_t(k_t)-p_tk_t = V_t(k'_t) - p_tk'_t + \lambda_t,$$

$$(8)\quad V'_t(k_t) - p'_tk_t = V'_t(k'_t) - p'_tk'_t - \lambda'_t,$$

where $\lambda_t \geqq 0$, $\lambda_t' \geqq 0$. Subtracting (8) from (7) and using $V_t(k_t') = -V_t'(k_t)$, as well as $V_t(k_t) = V_t'(k_t') = 0$, gives

(9) $(p_t'-p_t)(k_t'-k_t) = -(\lambda_t+\lambda_t') \leqq 0.$

Thus $L_p(t)$ is monotone increasing and bounded above. This line of argument leads once more to the conclusion of Theorem 9 with the condition that Assumption III holds for both paths and uniform value loss holds along one of them. To avoid contradiction with (9), the number of periods $N(\varepsilon)$ when $|k_t'-k_t| > \varepsilon$ cannot exceed $L_p(0)/\delta$.

We have proved

Theorem 9'. *Let* $\{k_t\}$, $\{k_t'\}$, $t = 0,1,\ldots$, *be optimal paths and assume* I, II, *and* III *for* $\{k_t\}$ *and assume* $k_0' \in$ *relative int* K_0. *Then support prices* $\{p_t\}$ *and* $\{p_t'\}$ *exist for* $\{k_t\}$ *and* $\{k_t'\}$ *respectively. Assume uniform value loss for either* (p_t,k_t) *or* (p_t',k_t'). *Then for any* $\varepsilon > 0$ *there is a number* $N(\varepsilon)$ *such that* $|k_t'-k_t| > \varepsilon$ *can hold for at most* $N(\varepsilon)$ *periods.*

However, Theorems 9 and 9' do not apply to objective functions that discriminate systematically against the future. The simplest of these, and one often used, is $u_t(x,y) = \rho^t u(x,y)$ where $0 < \rho < 1$ and u is a function independent of time that satisfies Assumptions I and II. Make Assumptions (G2) - (G5). Then (G1), or $u_t = u$ and $D_t = D$, all t, implies by Theorem 3 that an optimal stationary path $k_t = k$ exists. From the proof we find that k satisfies $u(k,k) \geqq u(x,x)$ for $(x,x) \in D$. Moreover, strict concavity at (k,k), provided by (G5) implies that k satisfying this maximizing condition is unique.

If ρ is now introduced, that is, the utility function $u_t'(x,y) = \rho^t u(x,y)$, $0 < \rho < 1$, is defined, for ρ sufficiently near 1, (G2) - (G4) will imply (S2) - (S4). Then for such a ρ, Theorem 8 implies that a stationary optimal path $k_t = k^\rho$ exists. From the proof, using Lemma 6, k^ρ satisfies $u(k^\rho,k^\rho)$ $\geqq u(x,y)$ for all $(x,y) \in D$ such that $\rho y-x \geqq (\rho-1)k^\rho$. Let $V(x)$ be the value function in the stationary model with $u_t = u$ and $u(k,k) = 0$. Let $K = \{x | V(x) > -\infty\}$. Assumptions (S3) and (S4) imply that K has an interior.

With Assumptions I, II, and (G2) - (G5) we can prove

Lemma 9. *For any* $\varepsilon > 0$ *there is* ρ' *such that* $|k^\rho-k| < \varepsilon$ *holds for some optimal stationary path,* $k_t = k^\rho$, *for all* t, *when* $1 > \rho > \rho'$.

By Theorem 8, Assumptions I, II, and (S1) - (S4) imply that a stationary optimal path $k_t = k^\rho$ exists. But for ρ' near ρ these assumptions are implied. Then, as mentioned above, such a path exists where $u(k^\rho,k^\rho)$ maximizes $u(z,w)$ over all (z,w) that satisfy $\rho w-z \geqq (1-\rho)k^\rho$, that is, over $(z,w) \in f(k^\rho,k^\rho)$. Let $\rho^s \to 1$, where $\rho' < \rho^s < 1$ and $s = 1,2,\ldots$. Since $|k^\rho| < \zeta$ by (G2), there is a subsequence (preserve notation) such that $k^{\rho^s} \to \bar{k}$. By Lemma 5, $f(x,x)$ is lower semi-continuous in $(x,x) \in \Delta$. Then the continuity of u and $u(k^{\rho^s},k^{\rho^s})$ maximal over $f(k^{\rho^s},k^{\rho^s})$ implies that $u(\bar{k},\bar{k})$ is maximal over $f(k^1,k^1)$ (Berge [1963], p. 116). Since $f(k^1,k^1)$ contains all $(x,x) \in D$, from Lemma 2 and the proof of Theorem 6 we find that $k_t = \bar{k}$ is a stationary optimal path when $\rho = 1$. Strict concavity of u near (k,k) from Assumption (G5), implies that a stationary optimal path k that satisfies $u(k,k) \geqq u(x,x)$ for $(x,x) \in D$ is unique. Thus $\bar{k} = k$ and the original sequence $k^{\rho^s} \to k$. For each value of ρ, $0 < \rho < 1$, choose k^ρ satisfying the condition of Lemma 6. Since $k^{\rho^s} \to k$ for an arbitrary sequence ρ^s, $s = 1,2,\ldots$, with $\rho^s \to 1$, we may conclude that $k^\rho \to k$ as $\rho \to 1$ and the lemma is proved.

With this preparation we can develop a turnpike theorem for the quasi-stationary model (Cass and Shell [1976]). Substitute $\rho^t u$ for u_t in (4) and (5) and multiply through by ρ^{-t}. Define current prices by $q_t = \rho^{-t}p_t$. Then we have

(11) $u(k_{t-1},k_t) + q_tk_t - \rho^{-1}q_{t-1}k_{t-1} = u(k_{t-1}',k_t') + q_tk_t' - \rho^{-1}q_{t-1}k_{t-1}' + \rho^{-t}\delta_t$

in place of (4) and a similar equation in place of (5). For each ρ, $0 < \rho < 1$, choose k^ρ satisfying the condition of Lemma 6. Let $k_t' = k^\rho$, all t, where k^ρ is the capital stock of the optimal stationary path and $p_t' = \rho^t q^\rho$ are the Weitzman prices provided by Theorem 8. Let $\{k_t(\rho)\}$ be an optimal path from $k_0 \in$ interior K and $p_t(\rho) = \rho^t q_t(\rho)$ the Weitzman support prices, also from Theorem 8. If u is strictly concave near k^ρ, for any $\varepsilon > 0$ $|k_{t-1}(\rho)-k^\rho| > \varepsilon$ implies there is $\delta > 0$ such that $\rho^{-t}\delta_t^\rho > \delta$. Put $\delta_t = \delta_t(\rho)$ and $\delta_t' = \delta_t^\rho$. In place of (6) we obtain

$$(12)\quad (q_t(\rho)-q^\rho)(k_t(\rho)-k^\rho) - \rho^{-1}(q_{t-1}(\rho)-q^\rho)(k_{t-1}(\rho)-k^\rho) = \rho^{-t}(\delta_t(\rho)+\delta_t^\rho),$$

for all $t \geqq 1$. Assumption (G5) implies that a neighborhood U of (k,k) exists within which u is uniformly concave. Suppose ρ' is chosen near enough to 1 so that every $(k^\rho,k^\rho) \in U$ for $1 > \rho > \rho'$. This is possible by Lemma 9. Then for $1 > \rho > \rho'$ and any $\varepsilon > 0$ there is $\delta > 0$ such that $|k_{t-1}(\rho)-k^\rho| > \varepsilon$ implies $\rho^{-t}(\delta_t(\rho) + \delta_t^\rho) > \delta$.

Suppose that the initial prices $q_0(\rho)$ for the path $k_t(\rho)$, and the prices q^ρ that support the stationary optimal path are bounded as $\rho \to 1$. Then $(q_0(\rho)-q^\rho)(k_0-k^\rho)$ is bounded as $\rho \to 1$, and ρ' may be selected near enough to 1 to imply, for $1 > \rho > \rho'$,

$$(13)\quad -(\rho^{-1}-1)(q_0(\rho)-q^\rho)(k_0-k^\rho) < \delta/2.$$

Let $L_c^\rho(t) = (q_t(\rho)-q^\rho)(k_t(\rho)-k^\rho)$. Then (12) and (13) imply $L_c^\rho(1) - L_c^\rho(0) > \delta/2$. Then $-(\rho^{-1}-1)L_c^\rho(1) < \delta/2$ also holds. So long as $k_\tau(\rho) - k^\rho > \varepsilon$, for $0 < \tau < t$, we may apply induction to obtain $L_c^\rho(t) - L_c^\rho(t-1) > \delta/2$, uniformly for $1 > \rho > \rho'$, or

$$(14)\quad (q_t(\rho)-q^\rho)(k_t(\rho)-k^\rho) - (q_{t-1}(\rho)-q^\rho)(k_{t-1}(\rho)-k^\rho) > \delta/2.$$

On the other hand, if we multiply through by ρ^{-t}, (9) becomes

$$(15)\quad (q_t(\rho)-q^\rho)(k_t(\rho)-k^\rho) = \rho^{-t}(\lambda_t+\lambda_t') \leqq 0.$$

We will see that $L_c^\rho(t) = (q_t(\rho)-q^\rho)(k_t(\rho)-k^\rho)$ may serve in place of $L_\rho(t)$ to prove a turnpike theorem, in the sense of convergence to a neighborhood of k^ρ, rather than to k^ρ itself.

First, we must show

Lemma 10. *The prices* $q_0(\rho)$ *and* q^ρ *are bounded as* $\rho \to 1$.

Maintain the normalization $u(k,k) = 0$. Let V_0^ρ be the value function at $t = 0$ when ρ is the discount factor. Then $V_0^\rho(k_0)$ is an increasing function of ρ which is bounded since $V(k_0)$ is finite. Consider the support formula

$$(16)\quad V_0^\rho(k_0) - q_0(\rho)\cdot k_0 \geqq V^\rho(x) - q_0(\rho)\cdot x,$$

implied by (4.12). Set $\bar{q}_0(\rho) = \dfrac{q_0(\rho)}{|q_0(\rho)|}$. If $|q_0(\rho)|$ is unbounded as $\rho \to 1$, there is a sequence $\rho^s \to 1$, $s = 1,2,\ldots$, such that $|q_0(\rho^s)| \to \infty$, and $\bar{q}_0(\rho^s) \to \bar{q}_0 \neq 0$. Then (16) implies

$$-\bar{q}_0\cdot k_0 \geqq -\bar{q}_0\cdot x,$$

for all $x \in K$, in contradiction to $k_0 \in$ int K. Therefore, $q_0(\rho)$ is bounded as $\rho \to 1$.

A similar argument works for q^ρ. According to Lemma 6

$$(17)\quad u(k^\rho,k^\rho) + \rho q^\rho k^\rho - q^\rho k^\rho \geqq u(z,w) + \rho q^\rho w - q^\rho z,$$

for all $(z,w) \in D$.

According to Lemma 9, $k^\rho \to k$ as $\rho \to 1$. If q^ρ is unbounded, choosing a subsequence and normalizing as before, we obtain in the limit, as a consequence of (17),

$$0 \geq \bar{q}(w-z),\ \bar{q} \neq 0,$$

for all $(z,w) \in D$, in contradiction to (G4). Thus q^ρ is also bounded as $\rho \to 1$.

Since $V_0^\rho(y)$ is a continuous function of y in K, it is bounded in any compact subset of the interior of K. Also for some $\varepsilon > 0$, for any $k_0 \in K$, there is an x such that $(k_0-x)_i > \varepsilon > 0$ for all i. Then (16) implies that the bound on $q_0(\rho)$ as $\rho \to 1$ is uniform for k_0 in a compact subset of int K. We have the

Corollary: *The prices $q_0(\rho)$ are uniformly bounded for k_0 in a compact subset of int* K.

The corollary implies that the support prices $q_t(\rho)$ along any path lying in a compact subset of int K are bounded as $t \to \infty$, since the $q_t(\rho)$ are possible choices of $q_0(\rho)$ for $k_0 = k_t$.

We may now prove the neighborhood turnpike theorem.

Theorem 10. *Assume* I *and* II. *Let* $u_t = \rho^t u$, $D_t = D$, *and assume* (G2) - (G5). *Also assume that the point* (k,k) *of* (G5) *satisfies* $k \in$ *int* D. *Let* $\{k_t\}$ *be an optimal path where* $k_0 \in$ *int* K. *Let* $\{k_t'\}$, $k_t' = k^\rho$, *all* t, *be an optimal stationary path given by Theorem* 8. *Then for any* $\varepsilon > 0$, *there is* $\rho(\varepsilon)$ *and* $N(\varepsilon)$ *such that* $1 > \rho > \rho(\varepsilon)$ *implies* $|k_t-k^\rho| < \varepsilon$ *holds for all* $t > N(\varepsilon)$.

Since the prices $q_0(\rho)$ and q^ρ are bounded as $\rho \to 1$ by Lemma 10, the argument preceding (15) may be applied. Let $N > -\frac{2L_c^\rho(0)}{\delta}$. Then (14) and (15) are inconsistent unless $|k_t(\rho)-k^\rho| < \varepsilon$ for some $t < N$. The choice of N is independent of ρ so long as $1 > \rho > \rho'$. This shows the optimal path must approach k^ρ at least once (see the "visit Lemma" of Scheinkman [1976]). However, we will show that there is a neighborhood of k^ρ in which the path remains thereafter.

By Lemma 9 and the assumption that $(k,k) \in$ int D, it is possible to choose ρ' so that $(k^\rho,k^\rho) \in C \subset$ int D for $1 > \rho > \rho'$, where C is compact. Then any y sufficiently near k^ρ will have $(y,y) \in$ int D, which implies $y \in$ int K, for any ρ with $1 > \rho > 0$. Let $U_\varepsilon = \{y \mid |y-k^\rho| \leq \varepsilon\}$. Choose ε so that $U_\varepsilon \subset$ int K for $(k^\rho,k^\rho) \in C$. Then the prices $q_t(\rho)$ are bounded for $k_t(\rho) \in U_\varepsilon$ for any ρ with $1 > \rho > \rho'$.

Uniformly bounded prices for $1 > \rho > \rho'$ and $k_t(\rho) \in U_\varepsilon$ also imply that L exists such that $0 \geq L_c^\rho(t) > L$ for $k_t(\rho) \in U_\varepsilon$. Then if $k_t(\rho) \in U_\varepsilon$, it follows from (12) that $L_c^\rho(t+1) \geq \rho^{-1}L$. However, $-L_c^\rho(t)$ is seen from (7) and (15) to be the sum of the remainder terms in the supports of the value function, thus $L \to 0$ as $|k_t(\rho)-k^\rho| \to 0$. Then for any $\varepsilon' > 0$, ε may be chosen so small that $L_c^\rho(t+1) \geq \rho^{-1}L$ implies $|k_{t+1}(\rho) - k^\rho| < \varepsilon'$. This follows from the strict concavity of u near (k^ρ,k^ρ) and thus of V_{t+1}^ρ, uniformly for ρ near 1. Also ε may be chosen small enough to insure

$$(18)\quad L_c^\rho(t+2) - L_c^\rho(t+1) \geq \delta + (\rho^{-1}-1)L > \delta/2.$$

Then $L_c(t+2) \geq \rho^{-1}L$ also holds and $|k_{t+2}(\rho) - k^\rho| < \varepsilon'$.

Let $U_{\varepsilon'} = \{x \mid |x-k^\rho| < \varepsilon'\}$. Then (18) implies that $k_{t+\tau}(\rho) \in U_{\varepsilon'}$, for $\tau \geq 2$, so long as $k_{t+\tau-1}(\rho)$ does not lie in U_ε. But $k_{t+\tau}(\rho)$ must eventually re-enter U_ε, or $L_c^\rho(t+\tau)$ will become positive, which is impossible. A repetition of the argument shows that $k_{t+\tau}(\rho)$ remains in $U_{\varepsilon'}$ again. Thus $k_{t+\tau}(\rho)$ can never leave $U_{\varepsilon'}$, and the theorem is proved.

Theorem 10 is weaker than Theorem 9 where $\rho = 1$, since it is not asserted that ρ can be chosen so that $k_t(\rho)$ converges asymptotically to k^ρ. Indeed, there may be other optimal stationary paths interior to D and cyclical paths as well (see Benhabib and Nishimura [1978]). However, the Assumption (G5) may be strengthened to give asymptotic convergence for ρ sufficiently near 1. Suppose that u has continuous second partial derivatives at (k,k) and the Hessian of u is negative definite there. Then ρ may be chosen near enough to 1 so that $Q(\rho) = \begin{bmatrix} \rho u_{11} & \rho u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ is negative definite in a neighborhood W of (k,k).

Since the neighborhood W expands as $\rho \to 1$, ρ may be chosen near enough to 1 to bring k^ρ inside W. But $Q(\rho)$ negative definite in a neighborhood of k^ρ implies, for a choice of ρ sufficiently near 1, that (14) will hold for some $\delta > 0$ for any $\varepsilon > 0$.

Indeed, write the left hand side of (14) as

$$(19)\quad L_c^\rho(t) - L_c^\rho(t-1) = -(u_2^t - u_2^\rho)(k_t - k^\rho) - (\rho u_1^t - u_1^\rho)$$

where $u_2^t = \frac{\partial}{\partial y} u(k_{t-1}, y)]_{y = k_t}$, $u_2^\rho = \frac{\partial}{\partial x} u(x, k^\rho)]_{x = k^\rho}$, and similarly for u_1^t and u_1^ρ.

We may express (19) in a small neighborhood of (k^ρ, k^ρ) as

$$(20)\quad u_{21}^\rho((k_{t-1} - k^\rho) + u_{22}^\rho(k_t - k^\rho))(k_t - k^\rho) + (\rho u_{11}^\rho(k_{t-1} - k^\rho) + u_{12}^\rho(k_t - k^\rho))(k_{t-1} - k^\rho) + o(\varepsilon) = L_c^\rho(t-1) - L_c^\rho(t).$$

where $o(\varepsilon)$ is of second order in ε and $\varepsilon = |k_{t-1} - k^\rho| + |k_t - k^\rho|$. If $Q(\rho)$ is negative quasi-definite, (20) implies

$$(21)\quad L_c(t) - L_c(t-1) = -\lambda\varepsilon + o(\varepsilon)$$

where λ = maximal characteristic root of $\frac{1}{2}(Q^T(\rho) + Q(\rho))$, $\lambda < 0$. Thus ε' may be chosen so that $-\lambda\varepsilon + o(\varepsilon) > -\frac{1}{2}\lambda\varepsilon$ for all $\varepsilon < \varepsilon'$. But from Theorem 10, ρ may be chosen near enough to 1 so that $|k_t(\rho) - k^\rho| < \varepsilon'$ for all $t > N(\varepsilon')$. Let $\bar{\rho}$ be such that $Q(\rho)$ is negative quasi-definite for $1 \geqq \rho > \bar{\rho}$ and $|k_t(\rho) - k^\rho| < \varepsilon'$ for all $t > N(\varepsilon')$ for $1 \geqq \rho > \bar{\rho}$. Then to avoid contradicting (15), $|k_t(\rho) - k^\rho| \to 0$ must hold. Indeed, for any $\varepsilon > 0$ there is $N_1(\varepsilon)$ such that $|k_t(\rho) - k^\rho| < \varepsilon$ for $t > N_1(\varepsilon)$ when $1 \geqq \rho \geqq \bar{\rho}$.

We have proved

<u>Theorem 10'</u>: *If in addition to the hypotheses of Theorem* 10, u *has continuous second partial derivatives at the optimal stationary path* (k,k) *of Assumption* 5 *and the Hessian of* u *is negative definite at* (k,k), *there is* $\bar{\rho}$ *such that* $1 > \rho > \bar{\rho}$ *implies for any* $\varepsilon > 0$ *there is* $N(\varepsilon)$ *such that* $|k_t - k^\rho| < \varepsilon$ *for all* $t > N(\varepsilon)$, *where* $\{k_t\}$ *is any optimal path satisfying* $k_0 \in \text{int } K$.

The fact that k_0 is assumed interior to E_+^n is not a restriction, since the capital stock space can be chosen differently for each t, so long as all stocks are included that can appear in that period given the initial stocks (McKenzie [1976]). Of course, the requirements that k_0 be interior to K and (k,k) be interior to D <u>are</u> substantive restrictions.

The corollary to Theorem 10 extends the classical theorem for $\rho = 1$ to the case $\rho < 1$ and sufficiently near 1. In the differentiable case this result was obtained by Scheinkman [1976]. A result of this type was also obtained by Brock and Scheinkman [1978]. Theorem 10 may be extended to utility functions u_t that depend more generally on time where a uniform concavity condition can be obtained in a way analogous to the move from $\rho^t u$ to u. Suppose there exist numbers $\rho_\tau > 0$ such that $\tilde{u}_t = \Pi_1^t \rho_\tau^{-1} u_t$ is uniformly strictly concave along $\{k_t\}$. Then the argument leading to Theorem 10 can be retraced in this broader context (McKenzie [1976]). A particular case would be that of variable discount factors, or $u_t = \Pi_1^t \rho_\tau u$ so that $\tilde{u}_t = u$ for all t.

In the special case of the stationary model another type of turnpike theorem was established in the course of proving Theorem 7 in section 6. It was shown there that even without strict concavity of u near a point (k,k) where sustainable utility is maximized, if this point is unique (Assumption (G5')), the average input-output vector of a maximal path $\frac{1}{T}\sum_{1}^{T}(k_{t-1},k_t)$ converges to (k,k). Brock [1970] refers to this behavior of maximal paths as an average turnpike property. The circumstances that underlie the average turnpike property become clearer when a general analysis of asymptotic behavior of maximal paths is made using the notion of the von Neumann facet in the following section.

The asymptotic properties of optimal paths in the continuous time model have been investigated along lines similar to those of this section, in particular, by Cass and Shell [1976] and Brock and Scheinkman [1976].

9. The von Neumann Facet

Although the support prices were found for maximal paths in section 4 with utility functions that were only assumed to be concave, the turnpike theorems that have been proved so far have used stronger assumptions involving strict concavity at least at an optimal path. Strict concavity is used to provide value losses $\delta_t(x,y) > 0$ whenever $(x,y) \neq (k_{t-1},k_t)$ for an optimal path $\{k_t\}$. However, if the basis for a value loss argument exists in terms of uniformity of concavity over time, it will still be true that paths must behave asymptotically to eliminate the value loss. This means that asymptotically optimal paths must be supported by the same prices. If we define a facet as the set of $(x,y) \in D_t$ that are supported by a particular price vector (p,q), the elimination of value losses will require that the input-output vectors of optimal paths approach the same facet. Thus a weaker form of convergence will continue to hold. This convergence may, in fact, lead to a turnpike in the original sense when the facets have an appropriate structure. This is a generalization of the turnpike theorems to the case where utility may not be strictly concave, and value losses do not necessarily appear off the turnpike.

The case of non-strictly concave utility is not really a borderline case in terms of the economic problem. Suppose that the extensive model has neo-classical production functions with homogeneous labor input and no net joint products. That is, if (x,y) is an input-output vector for the *jth* industry, $x_i \geqq y_i$ for $i \neq j$. Output is divided between consumption and terminal stocks. Let utility be a strictly concave function of consumption. Then the reduced model cannot have a strictly concave utility function in terms of initial and terminal stocks. A flat piece of the graph of $u_t(x,y)$, and thus a non-trivial facet, will be generated by the variations in activity levels which are consistent with the labor supply and with the consumption vector c that underlies $u_t(x,y)$. The possible variations will be significant whenever the variations of the input-output vector can be absorbed by the initial and terminal stocks without varying either c or the total labor supply. If the technology is irreducible in the sense that n activities must be used to obtain $y > x$ for $(x,y) \varepsilon D_t$, the dimension of the facet will be at least n-1 when stocks are increasing. To this degree input-output changes can be made to impinge only on the accumulation program.

Define $F_t(p,q)$ as all $(x,y) \varepsilon D_t$ such that

$$(1) \quad u_t(x,y) + qy - px = \sup(u_t(z,w) + qw - pz)$$

over $(z,w) \varepsilon D_t$. Concavity and closedness of u_t implies that $F_t(p,q)$ is a closed convex subset of D_t. Also F_t is an upper semi-continuous correspondence from $E_{t-1} \times E_t$ to the non-negative orthant of $E_{t-1} \times E_t$. Let

$$d((z,w),F_t) = \min|(z,w) - (x,y)|,$$

for $(x,y) \varepsilon F_t$. We reformulate the value loss result as (McKenzie [1968])

Lemma 11. *Let u_t satisfy Assumptions* I *and* II. *Let $F_t(p,q) \neq \phi$ be a facet of D_t. For any $\eta > 0$, $\varepsilon > 0$ there is $\delta > 0$ such that $|z| < \eta$ and $(z,w) \varepsilon D_t$ implies $\delta(z,w) > \delta$ for $d((z,w),F_t) > \varepsilon$.*

Consider a sequence (z^s,w^s) that violates the conclusion, that is, $|z^s| < \eta$, $d((z^s,w^s), F_t) > \varepsilon$, but $\delta_t(z^s,w^s) < \delta^s$ where $\delta^s \to 0$. By Assumption II there is a convergent subsequence whose limit $(\bar{z},\bar{w})$ satisfies $\delta_t(\bar{z},\bar{w}) = 0$, $|\bar{z}| \leqq \eta$, and $d((\bar{z},\bar{w}),F_t) \geqq \varepsilon$. However, $\delta_t(\bar{z},\bar{w}) = 0$ implies $(\bar{z},\bar{w}) \varepsilon F_t$, which is a contradiction.

Lemma 11 may be used to prove a theorem that is the analog of Theorem 9. Suppose that $\{k_t\}$, t = 0,1,..., is a maximal path and F_t is a sequence of facets

where $(k_{t-1},k_t) \varepsilon F_t$ for all t. Such a sequence is defined by the sequence of support prices $\{p_t\}$ guaranteed by Lemma 1. It is not unreasonable, in view of bounded labor services, to assume F_t to be bounded, even uniformly over time. Let us assume further that the value loss off F_t is uniform over t in the sense that η, ε, and δ may be chosen independently of t in Lemma 11. Let K_0 be the set of initial stocks with well defined values when $u(k_{t-1},k_t) = 0$, all t. Then the analog of the dual argument for Theorem 9' will prove convergence of maximal paths $\{k_t'\}$, from initial stocks $k_0' \varepsilon$ int K, to the facet sequence $\{F_t\}$. The argument for Theorem 9 is also valid for a primal version of convergence to the sequence of facets. If strict concavity holds, $F_t = \{k_{t-1},k_t\}$ and the original theorems are true. We may state

Theorem 11. *Let $\{k_t\}$, $\{k_t'\}$, t = 0,1,..., be maximal paths and assume* I *and* II, *and* III *relative to $\{k_t\}$. Let $\{p_t\}$ support $\{k_t\}$ and let $\{F_t\}$ be the corresponding facet sequence. If $k_0' \varepsilon$ relative int K_0 and there is uniform value loss along $\{F_t\}$, for any $\varepsilon > 0$ there is $N(\varepsilon)$ such that $d((k_{t-1}',k_t'),F_t) > \varepsilon$ can hold for at most $N(\varepsilon)$ periods.*

One case to which Theorem 11 applies is the stationary model of Section 6 with the strict concavity assumption (G5) omitted. Lemma 2 is valid since it does not use (G5), and there exists $p \geq 0$ such that $u(x,y) + py - px \leqq \bar{u}$ for all $(x,y) \varepsilon D$ where $\bar{u} = \max u(x,x)$ for $(x,x) \varepsilon D$. The price vector p defines a facet F(p,p). We may prove (Peleg [1973])

Lemma 12. *Under Assumptions* (G1) - (G4) *there is $(k,k) \varepsilon D$ such that $u(k,k) \geqq u(x,x)$, for all $(x,x) \varepsilon D$ and $k_t = k$, t = 0,1,..., is a maximal path.*

Let $C = \{(x,x)|u(x,x) = \bar{u}\}$. By Assumption I, C is compact. Then there is (k,k) such that

$$(2) \quad pk \leqq px \text{ for all } (x,x) \varepsilon C.$$

Suppose $k_t = k$ is not maximal. Then there is a path $\{k_t'\}$ and $T > 0$ such that $k_0' = k$ and

(3) $\sum_1^t (u(k_{t-1}', k_t') - \bar{u}) > \varepsilon > 0,$

for $t > T$. Consider $(\bar{k}_{t-1}, \bar{k}_t) = \frac{1}{t}\sum(k_{t-1}', k_t')$. Since k_t' is bounded by (G2), there is a point of accumulation $(\bar{k}, \bar{k})$ of the sequence $(\bar{k}_{t-1}, \bar{k}_t)$. Moreover, $u(\bar{k}_{t-1}, \bar{k}_t) \geqq \frac{1}{t}\sum u(k_{t-1}', k_t')$ by concavity. Then (3) implies $u(\bar{k}, \bar{k}) \geqq \bar{u}$, and $(\bar{k}, \bar{k}) \in C$. By Lemma 2, and (3), for $t > T$,

(4) $\varepsilon < \sum_1^t (u(k_{\tau-1}', k_\tau') - \bar{u} \leqq \sum_1^t p(k_{\tau-1} - k_\tau) = pk - pk_t.$

But $k_t \to \bar{k}$, so (4) implies $pk > p\bar{k} + \varepsilon$. This contradicts (2), since $(\bar{k}, \bar{k}) \in C$. Therefore, $k_t = k$ is a maximal path.

The set C and, in particular, the maximal path $k_t = k$, lies on the facet $F(p,p)$. We may set $F_t = F(p,p)$ in Theorem 11 and derive the convergence of k_t' to $F(p,p)$. In a similar way, Theorem 10 may be given a facet generalization where k^ρ is replaced by the facet $F^\rho = F(p^\rho, \rho p^\rho)$ on which (k^ρ, k^ρ) lies. The argument for Lemma 9 now proves that (k^ρ, k^ρ) converges to C, which is convex. It is clear that if a limit point k' of k^ρ is expansible, a dominating path from k' eventually becomes available to k^ρ as it converges to k', and becomes dominating for k^ρ as well as $\rho \to 1$, contradicting the optimality of k^ρ. Thus in this case at least the limit points of k^ρ must give rise to maximal stationary paths. Lemma 10 on the boundedness of $p_0(\rho)$ and p^ρ is valid for the generalization without change. Thus we have

<u>Theorem 12</u>. *Assume* I *and* II. *Let* $u_t = \rho^t u$, $D_t = D$, *and assume* (G2) - (G4). *Let* $\{k_t\}$ *be an optimal path where* $k_0 \in int$ K. *Let* $\{k_t'\}$, $k_t' = k^\rho$, *all* t, *be a*

stationary optimal path, and $p_t' = p^\rho$ *the support prices, given by Theorem 8. Then for any* $\varepsilon > 0$, *there is* $\rho(\varepsilon)$ *and* $N(\varepsilon)$ *such that* $1 > \rho > \rho'$ *implies* $d((k_{t-1}, k_t), F(p^\rho, \rho p^\rho)) > \varepsilon$ *holds for no more than* $N(\varepsilon)$ *periods.*

The principal change that must be made in the proof of Theorem 10 to obtain the proof of Theorem 12 is to replace the condition $|k_t(\rho) - k^\rho| > \varepsilon$ by the condition $d((k_{t-1}(\rho), k_t(\rho) - F(p^\rho, \rho p^\rho)) > \varepsilon$ whenever lower bounds are being deduced for value losses $\delta_t(\rho) + \delta_t^\rho$. Then the elimination of value losses to avoid contradiction with (8.15) forces convergence to the facet $F(p', \rho p^\rho)$ rather than to the path $k_t' = k^\rho$. The part of the earlier proof that required (k^ρ, k^ρ) to enter a strictly concave neighborhood of (k,k) is no longer needed, since it is no longer necessary to establish $F(p^\rho, \rho p^\rho) = (k^\rho, k^\rho)$.

Given further assumptions on the structure of the facets F_t of Theorem 11 to which the (k_{t-1}, k_t) belong, and which are defined by the (p_{t-1}, p_t), it may be that paths that remain close to the F_t for a long time must approach each other. This can be seen most easily for stationary or quasi-stationary models where one price supported path is a maximal stationary path supported by constant current prices so that $F_t = F$ for all t.

Choose points in the facet F that span the smallest flat containing F, say (x^i, y^i), $i = 1, \ldots, r$, where the dimension of F is $r-1 \leqq 2n$. Then any point $(z,w) \in F$ can be expressed as $\sum_1^r \alpha_i (x^i, y^i)$ where $\sum \alpha_i = 1$. If $\{k_t\}$ is a path on F, we have $\{k_{t-1}, k_t\} = \sum_1^r \alpha_i^t (x^i, y^i)$, and $(k_t, k_{t+1}) = \sum_1^r \alpha_i^{t+1} (x^i, y^i)$, or $\sum_1^r \alpha_i^t y^i = \sum_1^r \alpha_i^{t+1} x^i$. Suppose that $r = n+1$ and A and B are square matrices with columns $\begin{pmatrix} x^i \\ 1 \end{pmatrix}$ and $\begin{pmatrix} y^i \\ 1 \end{pmatrix}$, respectively. Then for each $t \geqq 0$, the equation $B\alpha^t = A\alpha^{t+1}$ must be satisfied for some vectors α^t if (k_{t-1}, k_t) lies on F. If A is non-singular, this may be written

(2) $\alpha^{t+1} = A^{-1}B\alpha^t$.

Suppose $A^{-1}B$ has only one characteristic root λ with absolute value one and this root is simple. Then $\lambda = 1$, since α must solve (2) where

$$\sum_1^r \alpha_i(x^i,y^i) = (k,k)$$

and k is the capital stock vector of the stationary maximal path. This path is, of course, optimal for the quasi-stationary case by Theorem 6. We will show later that it is optimal for the stationary case as well. If we make Assumption G2 of the stationary model of section 6 that sustainable stocks are bounded, $|k_t|$ is bounded by a number ζ. Then for any path $\{k_t\}$ on F, $k_t \to k$ must hold (McKenzie [1968]). This is easily seen if the characteristic roots are all simple, so the characteristic vectors span the complexification of the r-dimensional Euclidean space (Hirsch and Smale [1974], pp. 64-65). Then $k_t = \sum_1^r \alpha_i\lambda_i^t z^i$ where z^i is the characteristic vector associated with λ_i and α_i is a given number, possibly complex. If $|\lambda_i| > 1$, $\alpha_i = 0$ must hold, or else the path is unbounded as $t \to \infty$. If $|\lambda_i| < 1$, $\lambda_i^t \to 0$ as $t \to \infty$. Thus $k_t \to \alpha_1 z^1 = \alpha_1 k$, where $\lambda_1 = 1$. By an extension of this argument the same convergence property will be shown to hold for any path that converges to F. Then the convergence of maximal paths will once again be established.

The conditions needed for convergence of $\{k_t\}$ on F to k when a unique stationary optimal path exists are less stringent than nonsingularity of A suggests. Consider the point $(\bar{u},0)$ in W defined in section 6. Then $u + pv = 0$ for every $v = (y-x)$ and $(x,y) \in F$. Thus all (u,v) corresponding to points in F lie in a flat of dimension less than or equal to n in E^{n+1} and $(\bar{u},0)$ is expressible as a convex combination of no more than n+1 vectors $u^i,v^i) = (u(x^i,y^i), y^i-x^i)$. This means (u(k,k),k,k) is a convex combination of no more than n+1 vectors $(u(x^i,y^i),x^i,y^i)$ and the dimension of F may be expected to be less than or equal to n except for coincidence. Then A and B will have at least as many rows as columns and except for coincidence A will have linearly independent columns. Also the earlier argument can equally well be carried out using r × r submatrices of A and B. Moreover, small perturbations of the model will eliminate characteristic roots of absolute value one except for the root one which is present by construction. See Morishima [1969], chapters 10 and 13, for a careful discussion of the polyhedral case.

We may say that the structure of the von Neumann facet F is *stable* if, for any $\varepsilon > 0$, there is T such that every solution α^t of the difference equation (2), for which $(B\alpha^t, B\alpha^{t+1}) = (k_t', k_{t+1}') \in F$ for $t \geq 0$, satisfies $|k_t'-k| < \varepsilon$ for some k for all $t > T$ (Inada [1964]). The case outlined in the last paragraph is an example of a stable facet. Suppose that a bounded path $\{k_t'\}$ converges to F, but that $\{k_t'\}$ does not converge to k. Choose a sequence of neighborhoods U^s of F defined by $U^s = \{(x,y)|d(x,y),F) < \varepsilon^s > 0\}$ where $\varepsilon^s \to 0$. Let t_s be a sequence of times such that $(k_t',k_{t+1}') \in U^s$ for $t \geq t_s$. This is possible since (k_t',k_{t+1}') converges to F by assumption. Consider the sequence of paths $\{k_\tau^s\}$, $\tau = 0,1,\ldots$, where $k_\tau^s = k'_{t_s+\tau}$. Since $\{k_t'\}$ is bounded, we may use the Cantor process to choose a subsequence converging to a path $\{k_\tau''\}$. Then F closed implies that $(k_\tau'',k_{\tau+1}'') \in F$ for all $\tau \geq 0$. If $\{k_t'\}$ does not converge to k, given $n \geq 0$ the times t_s may be chosen so that $|k_n^s - k| > \varepsilon > 0$ for all s. This implies that there exist paths beginning at time t = 0 on F that lie outside an ε-neighborhood of k at time t = n where n may be set arbitrarily large, in contradiciton to the stability of F. We may prove

<u>Theorem 13</u>. *If a path* $\{k_t'\}$ *in a stationary or quasi-stationary model, satisfying Assumptions* I, II, *and* (G2), *converges to the von Neumann facet* F *and the structure of* F *is stable, then* $k_t' \to k$ *where* k *is the capital stock vector of the stationary optimal path. If* $\{k_t'\}$ *is a maximal path, it is optimal, in the quasi-stationary case, or if* $k_0' \in K$.

The convergence has been shown. Also $k_t = k$ must be optimal for the discounted case since it is price supported and bounded. However, the argument for optimality leading to Theorem 3 only uses (W1) and (W2), which are met here, together with the turnpike property which was proved using (W3). Since in the present case the turnpike property is established, optimality follows for $\{k_t\}$. To see that $\{k_t'\}$ is optimal. Let $\{k_t''\}$ be another path from the same initial stock, that is, $k_0' = k_0''$. By the support property for $k_t = k$ with $p_t = \rho^t p$, we deduce

$$(3)\quad \sum_1^T(u_t(k''_{t-1},k''_t) - u_t(k'_{t-1},k'_t) = \rho^T p(k'_T-k''_T) + \sum_1^T(\delta_t(k'_{t-1},k'_t) - \delta_t(k''_{t-1},k''_t))$$

If $0 < \rho < 1$, the first term on the right of (3) converges to 0. Since $\sum_1^\infty \delta_t(k'_{t-1},k'_t)$ is finite for $0 < \rho < 1$, there is a number $\lambda = \inf \sum_1^\infty \delta_t$ over all paths from k_0'. If $\{k_t'\}$ does not realize this minimum, some path $\{k_t''\}$ does better, and the limit of the left hand side, thus the left hand side of (3), is positive for this choice, so $\{k_t''\}$ overtakes $\{k_t'\}$ contradicting the maximality of $\{k_t'\}$. Thus $\sum_1^\infty \delta_t(k'_{t-1},k'_t) = \lambda$ and these limits are not positive. This implies $\{k_t'\}$ is optimal. This argument proves that any bounded maximal path is optimal when $0 < \rho < 1$.

If $\rho = 1$, $k_0' \in K$ implies that λ exists here too. If $k_t'' \not\to k$, $\sum \delta_t(k''_{t-1},k''_t) \to -\infty$ so $\{k_t''\}$ is not a rival for $\{k_t'\}$. However, $k_t'' \to k$ implies that the first term on the right converges to 0 once more, and the argument may be repeated.

The facet F where A is nonsingular and $A^{-1}B$ has a unique characteristic root with absolute value one, which is simple and equal to one, gives a particular case for Theorem 13. A condition which is equivalent to stability is that the stationary path on F be unique and there be no cyclic paths on F of constant amplitude. (McKenzie [1968]).

10. Differentiable Utility

If we assume, in addition to concavity and closedness of u_t, differentiability of u_t with respect to capital stocks, a new method of proving the turnpike theorem becomes available, due to Araujo and Scheinkman [1977], that does not depend on the condition that ρ be near 1. Differentiability also facilitates comparative studies analogous to the comparative statics of general equilibrium theory. The special assumptions which are used to obtain the results are the analogues in the dynamic setting of the familiar assumptions of comparative statics and stability theory for general equilibrium, that is, a dominant diagonal or negative definiteness for the appropriate Jacobian matrix (see Arrow and Hahn [1971], chapter 12). Negative definiteness is equivalent in the differentiable context to the value loss assumptions of the last section. However, the dominant diagonal assumption for the Jacobian matrix is independent of negative definiteness. The concavity of utility that is crucial for calculus of variations, and maximum theory in general, is still needed. This should not be surprising since the conditions of Weierstrass and Legendre in calculus of variations are *necessary* conditions for an optimal path.

Because of the differentiability of u_t we do not need to appeal to section 4 for support prices since the derivatives of u_t take their place. Recall that a path of accumulation is optimal if it catches up to every alternative path from the same initial stocks. If $\{k_t\}$ is a path of positive stocks, consider alternative paths $\{k_t'\}$ where $k_t' = k_t$ for $t \neq \tau$ and $k_\tau = x > 0$. Then $\{k_t\}$ catches up to $\{k_t'\}$ if and only if $u_\tau(k_{\tau-1},x) + u_{\tau+1}(x,k_{\tau+1}) \leqq u_\tau(k_{\tau-1},k_\tau) + u_{\tau+1}(k_\tau,k_{\tau+1})$. The differentiability assumption implies that this condition will be violated for an appropriate choice of x and τ unless

$$(1)\quad u_2^t(k_{t-1},k_t) + u_1^{t+1}(k_t,k_{t+1}) = 0,$$

for all t, where u_1^t denotes the vector of derivatives of u_t with respect to initial stocks and u_2^t the vector of derivatives with respect to terminal stocks. Thus (1) is a necessary condition for optimal paths and corresponds to the Euler condition of the calculus of variations.

We assume

(I') *The utility functions* $u_t(x,y) = \rho^t u(x,y)$, *where* $u(x,y)$ *is concave and closed on the convex set* D, *contained in the non-negative orthant of* E^{2n}. *Interior* D $\neq \phi$ *and* u *has continuous second partial derivatives in the interior of* D.

For the sake of simplicity we make our argument in terms of the quasi-stationary case $u_t = \rho^t u$, $0 < \rho < 1$, although the argument can be given in a general form applying to utility functions that depend on time in more complicated ways, reflecting changes in taste and technology (McKenzie [1977]). Let $\{k_t\}$, $t = 0,1,\ldots$, be a path satisfying (1) for $u_t = \rho^t u$, where the distance of the path from the boundary of $D_t = D$ is at least $\varepsilon > 0$ in all periods. Represent an arbitrary path $\{k_t'\}$ by z_t where $z_t = k_t' - k_t$, and rewrite (1), after dividing through by ρ^t, as

(2) $v_2(z_{t-1},z_t) + \rho v_1(z_t,z_{t+1}) = 0,$

for all t, setting $v(z_{t-1},z_t) = u(k_{t-1}',k_t')$ for all t. We will refer to $\{z_t\}$ also as a path. For a given $0 < \beta < 1$, let $x_t = \beta^{-t} z_t$. Let G_z be the set of paths $\{z_t\}$ with $|z_t| < \varepsilon/2$ and $\beta^{-t}|z_t| < \gamma < \infty$ for all t. Let G_x be the corresponding set of sequences $\{x_t\}$. Then G_x is contained in the Banach space ℓ_∞^n of bounded sequences of vectors in E_n. The norm $|x|_\infty$ of $x \in \ell_\infty^n = \sup |x_t|$ over $t \geq 0$, where $|x_t|$ is the Euclidean norm. The set G_x is not empty since it contains 0.

By the assumption that (k_{t-1}',k_t') is bounded interior to D, G_x has a non-empty interior in ℓ_∞^n.

We define a function ξ by

$$\xi(x_0,x)_t = \beta^{-t} v_2(\beta^{t-1}x_{t-1},\beta^t x_t) + \beta^{-t}\rho v_1(\beta^t x_t,\beta^{t+1}x_{t+1}),$$

$t = 0,1,\ldots$, where $x = \{x_t\}$, $t = 1,2,\ldots$. Then $\xi(0,0)_t = 0$ for all t, by the first order condition (2) for an optimum. If v^t has second partial derivatives at (0,0) that are bounded and uniformly continuous over t, $\xi(x_0,x)_t$ is bounded over t for small ε. Thus ξ maps F_x into ℓ_∞^n.

We will say that a path $\{k_t\}$ is *smooth* if it satisfies the Euler equations (1), is uniformly bounded from the boundary of D, and if u has second partial derivatives that are uniformly bounded from 0 and ∞ over t and uniformly continuous along $\{k_t\}$. It is possible to show for smooth paths that the derivative $D_x\xi$ at (0,0) is given by

$$(3)\quad \begin{aligned} (D_x\xi h)_1 &= (v_{22}^1 + \rho v_{11}^2)h_1 + \beta\rho v_{12}^2 h_2, \\ (D_x\xi h)_t &= \beta^{-1}v_{21}^t h_{t-1} + (v_{22}^t + \rho v_{11}^{t+1})h_t + \beta\rho v_{12}^{t+1}h_{t+1}, \end{aligned}$$

for $t = 2,3,\ldots$, where $h \in \ell_\infty^n$, $v_{ij}^t = v_{ij}(z_{t-1},z_t)$, and the partial derivatives are evaluated at (0,0). Also $D_x\xi$ is continuous at (0,0) (see Araujo and Scheinkman [1977]). We may represent $D_x\xi$ as an infinite matrix, or

$$(4)\quad D_x\xi = \begin{bmatrix} v_{22}^1 + \rho v_{11}^2 & \beta\rho v_{12}^2 & & & 0 \\ \beta^{-1}v_{21}^2 & v_{22}^2 + \rho v_{11}^3 & \beta\rho v_{12}^3 & & \\ & \beta^{-1}v_{21}^3 & v_{22}^3 + \rho v_{11}^4 & \beta\rho v_{12}^4 & \\ & & & \ddots & \\ 0 & & & & \end{bmatrix}$$

An infinite matrix M formed of $n \times n$ blocks M_{ij}, with M_{ii} invertible, is said to have dominant diagonal blocks if $\sup |M_{ii}^{-1}| < \infty$ over i and $\sup \sum_{j \neq i} |M_{ii}^{-1} M_{ij}| = \delta < 1$ over i. For a matrix argument $|\cdot|$ indicates the operator norm, that is $|M_{ij}|$ is $\sup |M_{ij} y|$ for $y \in E^n$, $|y| = 1$. M defines a transformation of ℓ_∞^n into ℓ_∞^n when $\sum_j |M_{ij}|$ is bounded over i. The boundedness of the second partial derivatives of u along a smooth path imply this condition for $D_x \xi$ if $\{k_t\}$ is regular. The matrix M is said to be *invertible* if it defines a linear homeomorphism of ℓ_∞^n onto ℓ_∞^n (Dieudonné [1960], p. 45). We may show

Lemma 13. *If an infinite matrix* M *that maps* ℓ_∞^n *into* ℓ_∞^n *has dominant diagonal blocks, it is invertible.*

Since M is bounded on the unit ball in ℓ_∞^n, it is a continuous linear map. Let M_1 be the matrix of diagonal blocks M_{ii} with 0's elsewhere. Since $|M_{ii}^{-1}|$ is bounded over i by the assumption of dominant diagonal blocks, M_1^{-1} exists and is continuous. Let $M_2 = M_1^{-1} M$. Then the dominant diagonal assumption implies $|M_2 - I| = \sup_i \sum_{j \neq i} |M_{ii}^{-1} M_{ij}| = \delta < 1$. Since $M_2 = I - (I - M_2)$, formally $M_2^{-1} = I + (I-M_2) + (I-M_2)^2 + \cdots$. But the Neumann series on the right hand side converges, so M_2 has a continuous inverse over ℓ_∞^n. Thus $M = M_1 M_2$ has a continuous inverse over ℓ_∞^n.

Assume that $D_x \xi$ has dominant diagonal blocks at (0,0), which corresponds to $k_t' = k_t$, for all t. This condition will hold for β sufficiently near 1 if it holds for $\beta = 1$. Then $D_x \xi$ is a linear homeomorphism of ℓ_∞^n onto ℓ_∞^n. Also $\xi(x_0, x)$ maps a neighborhood of (0,0) in $E^n \times \ell_\infty^n$ into ℓ_∞^n with $\xi(0,0) = 0$. We may apply the implicit function theorem (Dieudonné [1960], p. 265) to obtain a continuous function $\psi(x_0)$ valid in a neighborhood of $x_0 = 0$ such that $\xi(x_0, \psi(x_0)) = 0$ where $\psi(x_0)$ has continuous derivatives and $\psi(0) = 0$.

The continuity of ψ implies that $|x_0|$ may be chosen small enough to put x near 0, that is, $\sup |x_t| < \varepsilon$ over t for small positive ε. Then $z_t = \beta^t x_t$ for $\beta < 1$ implies that z_t converges exponentially to 0, that is k_t' converges exponentially to k_t. We note that for ε sufficiently small $\{k_t'\}$ is also a smooth path. This proves

Lemma 14. *If* (k_0, k) *is a path of accumulation that is smooth and the Jacobian of the map* ξ, *derived from the Euler equation* (1), *has dominant diagonal blocks, there is a neighborhood* W *of* k_0 *such that* $k_0' \in W$ *implies there is a smooth path* $\{k_t'\}$, $t = 0,1,\ldots$, *and* $k_t' \to k_t$ *exponentially as* $t \to \infty$.

In order to derive a local turnpike theorem from Lemma 14, it is only necessary to show that the paths $\{k_t'\}$ derived there are optimal paths from k_0' near k_0. An additional assumption is needed, which in the quasi-stationary case can take the form of (G2), introduced in section 6. The effect of (G2) is to bound any path. We can prove

Lemma 15. *If Assumptions* I', II, *and* (G2) *hold, any path that satisfies the Euler equation and is bounded interior to* D *is smooth, and any infinite smooth path is optimal.*

All that is needed to give smoothness for an Euler path that is bounded away from the boundary of D is that its second partial derivatives be bounded. However, smoothness is immediate by continuity of the derivatives if the path is confined to a compact subset of D. But this follows from (G2) and the fact that the path is bounded interior to D.

To show optimality for smooth paths observe that concavity of u implies for $\{k_t'\}$, $t = 0,1,\ldots,$

$$(5) \quad \rho^{t+1} u(k_t', k_{t+1}') + \rho^{t+1} u_2^{t+1} k_{t+1}' - \rho^{t+1} u_1^{t+1} k_t' \geq \rho^{t+1} u(x,y) + \rho^{t+1} u_2^{t+1} y - \rho^{t+1} u_1^{t+1} x$$

for $(x,y) \varepsilon D$, where $u_2^{t+1} = u_2(k'_t,k'_{t+1})$, for example. By the Euler equation (1), $u_2(k'_{t-1},k'_t) = \rho u_1(k'_t,k'_{t+1})$ for a smooth path $\{k'_t\}$. Thus $\rho^t u_2^t = \rho^{t+1}u_1^{t+1}$ in (5). Let $\rho_t = \rho^t u_2^t$, $t = 0,1,\ldots.$ Then (5) implies (W1) and the second part of (W2'). Since (G2) implies the second part of (W2'), $\{k'_t\}$ is optimal by Theorem 5.

Together Lemmas 14 and 15 imply

Theorem 14. *Suppose* $\{k_t\}$, $t = 0,1,\ldots$, *is a path that is smooth and satisfies the dominant diagonal condition, and Assumptions* I', II, *and* (G2) *are met by the utility function. Then every capital stock* k'_0 *near* k_0 *initiates a unique optimal path and this path converges exponentially to* $\{k_t\}$.

To see that the optimal path is unique, suppose there were a second optimal path $\{k''_t\}$. Consider a path $\{\bar{k}_t\}$ with $\bar{k}_t = \alpha k'_t + (1-\alpha)k''_t$, $0 < \alpha < 1$. Then $\alpha\sum_1^T u_t(k'_{t-1},k'_t) + (1-\alpha)\sum_1^T u_t(k''_{t-1},k''_t) = \sum_1^T u_t(\bar{k}_{t-1},\bar{k}_t) - \sum_1^T \varepsilon_t$, where $\varepsilon_t \geqq 0$. Thus the optimality of $\{k'_t\}$ and $\{k''_t\}$ implies that $\varepsilon_t = 0$, all t. For α sufficiently small $\{\bar{k}_t\}$ lies in a small neighborhood of $\{k'_t\}$ and thus of $\{k_t\}$. Since $\{\bar{k}_t\}$ is optimal it must satisfy the Euler equations. However, by the implicit function theorem the solution of the Euler equation in a small neighborhood of $\{k_t\}$ is unique. Thus $k''_t = k'_t$ for all t.

Theorem 14 is a local turnpike result. However, it may be used to prove a global theorem. Let H be the set of capital stocks, at t = 0, that initiate smooth paths along which the dominant diagonal condition is met. H is not empty if we have a smooth optimal path $\{k_t\}$ satisfying the dominant diagonal condition. In fact in the quasi-stationary model for $0 < \rho < 1$, and sufficiently close to 1, there is a stationary optimal path $k_t = k_0$, all t, by Theorem 8, and it is smooth if k_0 is interior to D. Then the dominant diagonal condition implies by Theorem 14 that any k'_0 in a neighborhood of k_0 initiates a smooth path $\{k'_t\}$ that converges to $\{k_t\}$. Moreover, the uniform continuity of the second partial derivatives near the path $\{k_t\}$ implies that the dominant diagonal condition is also met by $\{k'_t\}$ when the neighborhood is chosen small enough. Thus we may consider the maximal connected component C of H that contains k_0.

Let S be the subset of C such that the optimal path from $w \varepsilon S$ converges exponentially to $\{k_t\}$. If $w \varepsilon S$, Theorem 14 implies there is a neighborhood of w which is also in S. Let $\{k'_t\}$ be the optimal path from w and let y lie in this neighborhood. Then there is a path $\{k''_t\}$ from y, and $\beta < 1$, for which $|k_t-k''_t| \leqq |k_t-k'_t| + |k'_t-k''_t| \leqq \beta^t|k_0-w| + \beta^t|w-y|$, so k''_t also converges exponentially to k_t as $t \to \infty$. Thus S is open.

Now suppose that $x \varepsilon$ boundary S and $x \varepsilon C$. Then Theorem 14 applies and there is $y \varepsilon S$ near x such that the path $\{k''_t\}$ from y converges exponentially to the path $\{k'_t\}$ that departs from x. Therefore, k'_t must converge exponentially to k_t, or S is closed in C. But C is a connected set so S = C. We have proved a global result

Theorem 15. *Suppose* $\{k_t\}$, $t = 0,1,\ldots$, *is a path that is smooth and satisfies the dominant diagonal condition, and Assumptions* I', II, *and* (G2) *are met by the utility function. Let* H *be the set of capital stocks, at* t = 0, *that initiate smooth paths satisfying the dominant diagonal condition. Let* C *be the maximal connected component of* H *that contains* k_0. *Then* $x \varepsilon C$ *implies there is a unique optimal path* $\{k'_t\}$ *with* $k'_0 = x$ *and* $k'_t \to k_t$ *at an exponential rate, as* $t \to \infty$.

The crucial feature of the argument leading to the turnpike result is the invertibility of the derivative of the Euler functions. This derivative was used to define a transformation of ℓ_∞^n into ℓ_∞^n. Sometimes, however, other Banach spaces may be more effective. For example, if assumptions are made like those in section 8 to support a value loss argument, the appropriate space is Hilbert space ℓ_2^n. The invertibility lemma follows if the derivative is negative definite. Consideration of the matrix representation (4) shows that the derivative is negative definite if the matrix $\begin{bmatrix} \rho u_{11}^t & \rho u_{12}^t \\ u_{21}^t & u_{22}^t \end{bmatrix}$ is negative quasi-definite uniformly over the path. This is implied by uniformity over $D \cap W$, where $W = \{(x,y) \mid |x| < \max(|k_0|,\zeta)\}$ where ζ is from (S2). It has been shown (Brock and Scheinkman [1978]) that this condition implies the value loss assumption (8.14) used to prove Theorem 10. Thus the method of this section is very powerful for interior paths when u is twice continuously differentiable.

The arguments used in this section like those in section 9 are not limited to the quasi-stationary case. With minor complications they can be adapted to utility functions $u_t(x,y)$ which depend on time in the way described in section 2 (McKenzie [1977]).

11. Comparative Dynamics

By use of the infinite Jacobian matrix of the first order conditions (the discrete Euler conditions) for an optimal path it is possible to derive comparative dynamic results for the differentiable model (Araujo-Scheinkman [1978]). These are analogous to the comparative static results proved in general equilibrium theory and use the same assumptions adapted to the infinite case. The Jacobian matrix is shown to be negative definite, or it is assumed to have dominant diagonal blocks with certain sign patterns for diagonal and off-diagonal blocks. The parameters that shift demand between the numéraire and other goods in the static case are replaced by the discount factor or the initial stocks in the dynamic case.

Let $\{k_t\}$, $t = 0,1,\ldots$, be an optimal path. Let $z_t = k_t' - k_t$, and $z = \{z_t\}$, $t = 1,2,\ldots$. Define $\zeta(z_0,z,\rho)$ for $0 < \rho < 1$ by

$$(1) \quad \zeta_t(z_0,z,\rho) = v_2(z_{t-1},z_t) + \rho v_1(z_t,z_{t+1}),$$

where $v(z_{t-1},z_t) = u(k_{t-1},k_t)$. If $\{k_t\}$ is a regular path and B_z is the set of paths $\{z_t\}$ with $|z_t| < \varepsilon$, for small ε, ζ maps B_z into ℓ_∞^n. Similarly if H_z is the set of paths $\{z_t\}$ with $\sum_1^\infty |z_t|^2 < \infty$ and $|z_t| < \varepsilon$, for small ε, ζ maps H_z into ℓ_2^n. In the first case ζ maps a neighborhood of 0 in ℓ_∞^n into ℓ_∞^n, and in the second case ζ maps a neighborhood of 0 in ℓ_2^n into ℓ_2^n.

As in section 10, under Assumption I$'$ for a smooth path $\{k_t\}$, $D_z\zeta(0,0,\rho)$ can be represented by an infinite matrix, in either space,

$$(2) \quad D_z\zeta = \begin{bmatrix} v_{22}^1 + \rho v_{11}^2 & \rho v_{12}^2 & & & 0 \\ v_{21}^2 & v_{22}^2 + \rho v_{11}^3 & \rho v_{12}^3 & & \\ & v_{21}^3 & v_{22}^3 + \rho v_{11}^4 & \rho v_{12}^4 & \\ & & & \ddots & \\ 0 & & & & \end{bmatrix}.$$

In this expression $v_{ij}^t = v_{ij}(k_{t-1},k_t)$. Suppose the quadratic bilinear form $h^T(D_z\zeta)h$ is negative definite, that is, $h^T(D_z\zeta)h < -\varepsilon|h|$, for all $h \in \ell_2^n$ and some $\varepsilon > 0$. Then $D_z\zeta$ is invertible on ℓ_2^n (Araujo and Scheinkman [1977], p. 619). It is clear from the representation (2) that $D_z\zeta$ will be negative definite if $\begin{bmatrix} \rho v_{11}^t & \rho v_{12}^t \\ v_{21}^t & v_{22}^t \end{bmatrix}$ is negative quasi-definite, uniformly with respect to t along the path $\{k_t\}$. At the stationary optimal path, $k_t = k$, all t, $D_z\zeta$ is negative definite if and only if $\begin{bmatrix} \rho v_{11} & \rho v_{12} \\ v_{21} & v_{22} \end{bmatrix}$ is negative quasi-definite, where $v_{ij} = u_{ij}(k,k)$. Also, from Lemma 13, $D_z\zeta$ is invertible over ℓ_∞^n when the dominant diagonal condition is met. These are the two conditions which have been shown to imply a turnpike theorem. As in the general equilibrium tâtonnement, there is a close relationship between conditions which imply stability and conditions which allow comparison of equilibrium paths.

As for $D_z\xi$ in section 8, the invertibility of $D_z\zeta$ allows the implicit function theorem to be applied to obtain a function $\phi(z_0,\rho')$, defined in some small neighborhood of $(0,\rho)$, such that $\zeta(z_0,\phi(z_0,\rho'),\rho') = 0$. Also $\phi(z_0,\rho')$ is differentiable and the derivatives are given by

(3) $D_{z_0}\phi(z_0,\rho') = -[D_z\zeta(z_0,\phi(z_0,\rho'),\rho')]^{-1}\cdot D_{z_0}\zeta(z_0,\phi(z_0,\rho'),\rho')$, and

$$D_\rho\phi = -[D_z\zeta]^{-1}\cdot D_\rho\zeta,$$

Dieudonné [1960], p. 268. We first show

Lemma 16. *If $D_z\zeta$ is invertible on ℓ_∞^n, Assumption* I' *implies*

(4) $\sum_1^\infty \rho^t[\frac{dz}{d\rho}]_t[D_\rho\zeta]_t \geqq 0,$

where $D_z\zeta$ and $D_\rho\zeta$ are evaluated at $(z_0,\phi(z_0,\rho'),\rho')$, $\frac{dz}{d\rho} = D_\rho\phi(z_0,\rho')$, *and* (z_0,ρ') *is sufficiently near* $(0,\rho)$.

Since $\frac{dz}{d\rho} = -[D_z\zeta]^{-1}\cdot D_\rho\zeta$ by (3), we obtain $D_z\zeta\cdot\frac{dz}{d\rho} = -D_\rho\zeta$. Therefore,

(5) $\sum_1^\infty \rho^t[\frac{dz}{d\rho}]_t[D_z\zeta\cdot\frac{dz}{d\rho}]_t = -\sum_1^\infty \rho^t[\frac{dz}{d\rho}]_t[D_\rho\zeta]_t.$

The left hand side of (5) is equal to $[\frac{dz}{d\rho}]^T A\frac{dz}{d\rho}$, where A is equal to the matrix obtained from $D_z\zeta$ when the *tth* row is multiplied by ρ^t. However, it is easily seen from (2) that A is negative semi-definite if $\begin{bmatrix} v_{11}^t & v_{12}^t \\ v_{21}^t & v_{22}^t \end{bmatrix}$ is negative semi-definite, which is implied by the concavity of v. The concavity of v is immediate from the concavity of u given by Assumption I'. Also the convergence of the sums in (5) follows from the fact that the derivatives belong to ℓ_∞^n and $0 < \rho < 1$. Of course, $D_z\zeta$ is a invertible on ℓ_∞^n when the path $\{k_t\}$ is smooth and the dominant diagonal condition holds.

From (1) we observe that $[D_\rho\zeta(0,0,\rho)]_t = v_1(0,0) = u_1(k_t,k_{t+1}) = \rho^{-1}u_2(k_{t-1},k_t)$. Thus $([D_\rho\zeta]_t,\rho[D_\rho\zeta]_{t+1})$ supports $u(k_t,k_{t+1})$ in the sense of (4.12) by virtue of the concavity of u. Put $p^t = \rho^t[D_\rho\zeta]_t$, $t = 1,2,\ldots$ and $p^0 = u_1(k_0,k_1)$. Then $\{p^t\}$, $t = 0,1,\ldots$, satisfies (4.12). Moreover, by the differentiability of u, these supports are unique, so they must satisfy (4.11) as well by Lemma 1. Since $\frac{dz}{d\rho} = \frac{dk}{d\rho}$, the conclusion of Lemma 16 may be written $\sum_{1}^{\infty} p^t \frac{dk_t}{d\rho} \geqq 0$, or an increase in the discount factor for utility cannot reduce the present value of the stream of capital stocks at the support prices.

The conclusion of Lemma 16 holds equally well when $D_z\zeta$ is invertible on ℓ_2^n by the same argument. As mentioned earlier $D_k\zeta$ will be invertible for a regular path under Assumption I" if the quadratic bilinear form $h^T(D_k\zeta)h$ is negative definite, that is, if $\begin{bmatrix} \rho v_{11}^t & \rho v_{12}^t \\ v_{21}^t & v_{22}^t \end{bmatrix}$ is uniformly negative quasi-definite with respect to t. Indeed, in this case it is unnecessary to multiply by ρ^t. Current prices may be used, that is,

$$(6)\quad \sum_{1}^{\infty}[\tfrac{dz}{d\rho}]_t[D_\rho\zeta]_t = \sum_{1}^{\infty} q_t[\tfrac{dz}{d\rho}]_t > 0$$

will hold. However, the economic meaning of a sum of current values is not clear.

The results so far are not intrinsic to the stationary model. However, for the stationary model when a stationary optimal path exists that is interior, Araujo and Scheinkman [1978] have shown a more intimate connection between stability and Lemma 16. If the linear approximation to the Euler equations, as a system of difference equations, is asymptotically stable at the stationary optimal path, and the optimal path $\{k_t\}$ converges to the stationary optimal path, then the Jacobian matrix $D_z\zeta$ along this path is invertible on ℓ_∞^n, and the consequence (4) may be drawn. A path that converges in this fashion is said to satisfy a strong global turnpike condition.

The foregoing discussion may be collected in

Theorem 16. *Assume* I', II, *and* (G2), *and let* $\{k_t\}$, $t = 0,1,\ldots$ *be a regular optimal path. Let* $\{p^t\}$ *be the unique support prices for* $\{k_t\}$. *Then* $\sum_{1}^{\infty} p_t \frac{dk_t}{d\rho} \geqq 0$ *if any of the following conditions hold*

(1) *The Jacobian* $D_z\zeta(0,0,\rho)$ *has dominant diagonal blocks along* $\{k_t\}$, *where* $z_t = k_t' - k_t$.

(2) *The matrix* $\begin{bmatrix} \rho u_{11}^t & \rho u_{12}^t \\ u_{21}^t & u_{22}^t \end{bmatrix}$ *is negative quasi-definite along* $\{k_t\}$, *uniformly with respect to* t.

(3) *The path* $\{k_t\}$ *satisfies a strong global turnpike condition.*

If condition (2) *holds, the inequality is strict and* p_t *may be replaced with* $q_t = \rho^{-t}p_t$.

It should be noted that Theorem 16 does not make a comparison of stationary optimal paths. Even when $k_t = k_0, p_t = p_0$, for all t, we cannot expect $\frac{dk_t}{d\rho}$ to be constant with respect to t. A shift in ρ to ρ' will lead to a new *stationary* optimal path $k_t' = k_0'$, and the new *optimal* path from k will converge to k_0'.

Comparative dynamic results may also be obtained when the initial stocks vary if appropriate assumptions are made on the signs of elements of the Jacobian. These assumptions will be sufficient to sign the inverse of the Jacobian matrix just as in the static case of general equilibrium theory. The crucial mathematical tool is a generalization to infinite dimensions of the theorem on non-negative inverses for Leontief type matrices. Araujo and Scheinkman [1978] proved

Lemma 17. *Let* M *be an infinite matrix written as a collection of* n × n *blocks* M_{ij}, i,j = 1,2,..., *with* sup $\sum_{j=1}^{\infty}|M_{ij}| < \infty$, *over* i. *If* M *has dominant diagonal blocks and* $M_{ii}^{-1} \leqq 0$, $M_{ij} \geqq 0$, *for* $i \neq j$, *then* $M^{-1} \leqq 0$.

As in proving Lemma 13, let M_1 be the matrix of diagonal blocks M_{ii} with 0's elsewhere. Let $M_2 = M_1^{-1}M$. As before $M_2^{-1} = I + (I-M_2) + (I-M_2)^2 + \ldots$. Since $M_1^{-1} \leqq 0$ and $M_{ij} \geqq 0$ for $i \neq j$, $I - M_2 \geqq 0$. Thus $M_2^{-1} \geqq 0$ and $M^{-1} = M_2^{-1}M_1^{-1} \leqq 0$.

The condition $M_{ii}^{-1} \leqq 0$ will be satisfied by the theorem for Leontief matrices (McKenzie [1960]) if M_{ii} has quasi-dominant diagonal elements, either by rows or columns, that are negative, and the off-diagonal elements are non-negative. A square matrix A has *quasi-dominant diagonal elements* by rows if there exist numbers $d_i > 0$ such that $d_i|a_{ii}| > \sum_j d_j|a_{ij}|$ for all i, *mutatis mutandis* for columns.

Assume that the Jacobian matrix (2) of the Euler conditions $D_z\zeta$ satisfies the conditions of Lemma 17 on an optimal path. That is, $(v_{22}^t + \rho v_{11}^{t+1})^{-1} \leqq 0$ and $v_{12}^{t+1} \geqq 0$, $v_{21}^{t+1} \geqq 0$, for t = 1,2,.... Then if $[D_z\zeta]^{-1}$ exists, it will satisfy $[D_z\zeta]^{-1} \leqq 0$. However, by (3), $D_{z_0}\, \phi\, (z_0,\rho') = -[D_z\zeta]^{-1}\cdot D_{z_0}\zeta$. From (1), $[D_{z_0}\zeta]_1 = v_{21}(z_0,z_1)$, and $[D_{z_0}\zeta]_t = 0$, for t > 1. Thus, by assumption, $D_{z_0}\zeta \geqq 0$, and finally $D_{z_0}\, \phi\, (z_0,\rho') = \frac{dk}{dk_0} \geqq 0$. The effect of increasing any initial stock is to cause all subsequent stocks along the optimal path to increase or remain constant. This justifies

<u>Theorem 17</u>. *Assume* I', II, *and* (G2), *and let* $\{k_t\}$, t = 0,1,..., *be a smooth optimal path. Suppose the matrix* $D_z\zeta(0,0,\rho)$ *has dominant diagonal blocks, and the sign conditions* $(u_{22}^t + \rho u_{11}^{t+1})^{-1} \leqq 0$, $u_{12}^{t+1} \geqq 0$, $u_{21}^{t+1} \geqq 0$, t = 1,2,..., *are met by the utility functions* $u^t = u(k_{t-1},k_t)$. *Then* $dk_t/dk_0 \geqq 0$ *for all* t.

12. Comparative Statics

Comparative statics is confined to the stationary or quasi-stationary model and compares optimal stationary paths which correspond different values of the discount factor or other parameters of the model. Our interest will lie in the quasi-stationary model where the following assumption holds:

(I") *The utility function* $u_t = \rho^t_u$ *for* $0 < \rho \leqq 1$ *and* u *is concave and closed over* D *contained in the non-negative orthant of* E^{2n}. *Interior* $D \neq \phi$. *Also there is an optimal stationary path* $\{k_t\}$ *interior to* D *with* $k_t = k$, *where* u(k,k) *has continuous second partial derivatives and the Hessian matrix is negative definite.* We will be concerned with the effect of small changes in parameters for optimal stationary states (k,k) which are interior to D.

Let (k,k) be an optimal stationary state interior to D. Then the first order conditions for optimality (10.1) imply

(1) $u_2(k,k) + \rho u_1(k,k) = 0.$

As noted in section 11, if the matrix $Q(\rho) = \begin{bmatrix} \rho u_{11} & \rho u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ is negative quasi-definite the local turnpike theorem holds, that is, for any capital stocks in a small neighborhood of k, the unique optimal path converges to $k_t = k$, as $t \to \infty$. Then we may say that the optimal stationary state (k,k) is locally stable.

The Jacobian matrix of (1) with respect to k is $J(\rho) = u_{21} + u_{22} + \rho u_{11} + \rho u_{12}$. If this matrix is non-singular, the implicit function theorem may be applied to (1). That is, if (k',ρ') satisfy (1) for $0 < \rho' \leqq 1$ and $(k',k') \in$ interior D, there is a unique differentiable function $k(\rho)$ such that $(k(\rho),\rho)$ satisfy (1) for ρ near ρ', and $k(\rho') = k'$. Then we may consider the inequality

(2) $q \cdot \frac{dk(\rho)}{d\rho}]_{\rho=\rho'} > 0.$

If (2) is satisfied at (k',ρ') Burmeister and Turnovsky [1972] say that the model is *regular* at (k',ρ').

If the necessary condition (1) for optimality is totally differentiated with respect to ρ, we obtain

(3) $$(u_{21} + u_{22} + \rho u_{11} + \rho u_{12})\frac{dk}{d\rho} + u_1 = 0, \text{ or}$$
$$J(\rho)\frac{dk}{d\rho} = -u_1 = -q$$

Multiplying (3) by $\frac{dk}{d\rho}$ on the left gives

(4) $$(\frac{dk}{d\rho})^T J(\rho)\frac{dk}{d\rho} = -q \cdot \frac{dk}{d\rho}.$$

But if $Q(\rho)$ is negative quasi-definite, so is $J(\rho)$ and (4) implies (2). Thus we have

Theorem 18. *Under Assumption* I", *the sufficient condition for local stability of an optimal stationary state,* $Q(\rho)$ *negative quasi-definite, implies that the stationary state is regular.*

Another condition that implies $J(\rho)$ negative quasi-definite and thus establishes regularity is $u_{21} + u_{22}$ negative quasi-definite. Put $J(\rho,\alpha) = u_{21} + u_{22} + \alpha u_{11} + \alpha u_{12}$, evaluated at $k(\rho)$. Then $J(\rho,\alpha)$ is negative quasi-definite when $\alpha = 0$, and $J(\rho,\alpha)$ is negative definite for $\alpha = 1$. Since $J(\rho) = \rho(J(\rho,1)) + (1-\rho)(J(\rho,0))$, $J(\rho)$ is negative quasi-definite for $0 < \rho < 1$ (Dasgupta [1978]). The analogous condition in the continuous time model is shown by Magill [1977] to imply stability for that model. An interpretation of $u_{21} + u_{22}$ is the effect on the marginal utility cost of investment of an increase in initial stock with investment held constant. If we write $x_{t+1} = k_{t+1} - k_t$ and $U(k_t,x_{t+1}) = u(k_t,k_{t+1})$, then $u_{21} + u_{22} = U_{21}$.

According to Theorem 14, local asymptotic stability holds around an optimal path if the assumption of dominant diagonal blocks is met along this path. For an optimal stationary path the dominant diagonal assumption for the infinite Jacobian matrix $D_x\xi(x_0,x)$ with $B = 1$ is reduced to

(5) $$|(u_{22} + \rho u_{11})^{-1}(u_{21} + \rho u_{12})| < 1,$$

since the non-zero blocks of each row are the same, with $u^t_{ij} = u_{ij}(k,k)$, all t, and i, j = 1,2. Recall that $|\cdot|$ denotes the operator norm, that is $|A| = \sup|Ax|$ for $|x| = 1$. As in the case of market tâtonnement, the dominant diagonal assumption is not effective by itself but requires a supplement, for example, symmetry or sign restrictions on the elements. Assume, as in the dynamic case of Theorem 17, that $(u_{22} + \rho u_{11})^{-1} \leqq 0$ and $u_{12} = u^T_{21} \geqq 0$. From (3) we have

(6) $$\frac{dk}{d\rho} = -(I + (u_{22} + \rho u_{11})^{-1}(u_{21} + \rho u_{12}))^{-1}(u_{22} + \rho u_{11})^{-1}q.$$

By the argument of Lemma 17 and the sign assumptions $(I + (u_{22} + \rho u_{11})^{-1}(u_{21} + \rho u_{12}))^{-1}$ exists and is non-negative. Therefore $\frac{dk}{d\rho} = Mq$, where $M \geqq 0$. Since (k,k) interior to D implies that q is positive, we have $\frac{dk_i}{d\rho} > 0$, for all i, or an increase in the discount factor leads to an increase in all capital stocks. This justifies

Theorem 19. *Under Assumption* I", *if the dominant diagonal block condition holds and* $(u_{22} + \rho u_{11})^{-1} \leqq 0$, $u_{12} \geqq 0$, *the optimal stationary state satisfies* $\frac{dk_i}{d\rho} > 0$, *all* i.

Of course, Theorem 19 implies regularity of the stationary state but it is much stronger than regularity. Also the condition $(u_{22} + \rho u_{11})^{-1} \leqq 0$ is implied, as noted earlier, if $(u_{22} + \rho u_{11})$ has quasi-dominant diagonal elements, by row or by column, and the off-diagonal elements are non-negative.

The relation between stability and regularity, illustrated by Theorems 18 and 19, seems to be widespread, that is, sufficient conditions for local stability often imply regularity of the stationary state, whether the discrete time or the continuous time model is used. For the continuous time model additional examples may be found in Brock [1976]. Results of this type illustrate the *Correspondence Principle* of Samuelson, that "the problem of stability of equilibrium is intimately tied up with the problem of deriving fruitful theorems in comparative statics" (Samuelson [1947]). Also see Burmeister and Long [1977].

For further examples of the Correspondence Principle we may consider the autonomous difference equation of second order

$$(7)\quad u_2(k_{t-1},k_t) + \rho u_1(k_t,k_{t+1}) = 0.$$

This is the form taken by the necessary condition of optimality (8.1) for a stationary model. It is approximated in a small neighborhood of the optimal stationary state (k,k) by the linear equation

$$(8)\quad u_{21}z_{t-1} + (u_{22} + \rho u_{11})z_t + \rho u_{12}z_{t+1} = 0,$$

where $z_t = k_t - k$. The characteristic equation of (8) is

$$(9)\quad \det\,(u_{21} + (u_{22} + \rho u_{11})\lambda + \rho u_{12}\lambda^2) = 0,$$

where det A is the determinant of A. Suppose (9) has n roots of absolute value less than 1 and $\det u_{21} \neq 0$. Araujo and Scheinkman [1977] have shown that these assumptions imply the local turnpike theorem for (k,k), or local stability for optimal paths near (k,k). We will show that, if u_{21} is also symmetric, (k,k) is regular. This argument is due to Dasgupta [1978].

Rewrite (9) as

$$(10)\quad \det\,(A + B\lambda + \rho A\lambda^2) = 0.$$

The proof that (k,k) is regular depends on

<u>Lemma 18</u>. *If A is non-singular, the characteristic roots of $B^{-1}A$ are less than $\frac{1}{1+\rho}$ in absolute value, if and only if there are* **n** *roots of* (10) *with absolute value less than* 1.

Since -B is positive definite and A is symmetric there is a non-singular matrix P such that $P^TBP = -I$ and $P^TAP = -R$ where R is a diagonal matrix with the characteristic roots of $B^{-1}A$ on the diagonal. Also R is real. See Gantmacher [1960], p. 310. Since A is non-singular, the diagonal elements r_i of R are non-zero.

(10) is equivalent to $\det\,(P^T(A + B\lambda + \rho A\lambda^2)P) = 0$, or $\det\,(\lambda I + (1+\rho\lambda^2)R) = 0$. Thus the roots of (10) are the roots of the equations

$$(11)\quad \lambda + (1+\rho\lambda^2)r_i = 0,\quad i = 1,\ldots,n,$$

where repeated roots are counted. The discriminant of (11) is $(1 - 4\rho r_i^2)$. Thus the ith root is real if $|r_i| \leqq \frac{1}{1+\rho}$.

Suppose all $|r_i| < \frac{1}{1+\rho}$. Then all roots are real. Also $|A| \neq 0$ implies $\lambda = 0$ is not a root of (10). Then (11) implies

$$(12)\quad \rho|\lambda| + \frac{1}{|\lambda|} \geqq \frac{1}{|r_i|} > 1 + \rho.$$

This gives $(|\lambda|-1) > (|\lambda|-1)\frac{1}{\rho|\lambda|}$. It is clear from substitution in (11) that λ a root implies $\frac{1}{\rho\lambda}$ is the other root. Therefore, at least one of the roots has absolute value less than 1. Since this is true for all i, there must be n roots λ_i of (10) with $|\lambda_i| < 1$.

On the other hand, suppose there are n roots λ_i of (10) with $|\lambda_i| < 1$. If the roots λ_i are real, (12) implies that $|r_i| = \frac{1}{1+\rho}$ for $|\lambda| = 1$. Also the derivative of the left hand side of (12) with respect to $|\lambda|$ is negative for $|\lambda| < 1$. Then $|r_i| < \frac{1}{1+\rho}$ for $|\lambda| < 1$. If a root λ_i is complex, it follows that $\frac{1}{\rho\lambda} = \bar{\lambda}$, or $\frac{1}{\rho} = |\lambda| > 1$ in contradiction to the hypothesis. Thus $|r_i| < \frac{1}{1+\rho}$ holds for all i.

To show regularity, assume there are n roots λ_i of (10) with $|\lambda_i| < 1$. By Lemma 18 if r is any root of $B^{-1}A$, then $\frac{-1}{1+\rho} < r < \frac{1}{1+\rho}$. But -B positive definite and A symmetric implies $\min_i r_i \leq \frac{x^TAx}{x^TBx} \leq \max_i r_i$, where $x \neq 0$, and $r_i, i = 1,\ldots,n$, are the roots of $B^{-1}A$ (Gantmacher [1960], p. 319). Thus $|\frac{x^TAx}{x^TBx}| < \frac{1}{1+\rho}$, or, since B is negative definite, $x^T(A + B + \rho A)x = x^T(u_{21} + u_{22} + \rho u_{11} + \rho u_{12})x < 0$ for $x \neq 0$. By (4) this implies regularity for (k,k).

The results of Araujo and Scheinkman also show that if (8) has n+1 roots of absolute value larger than 1, the stationary state cannot be stable. Thus our result may be stated

Theorem 20. *Assume* I" *and let* (8) *represent the Euler equations linearized about the optimal stationary path* $k_t = k$. *If* (8) *has no roots of absolute value equal to* 1, *and* u_{12} *is symmetric, the optimal stationary state is regular if it is locally stable.*

The symmetry of u_{12} is equivalent to the symmetry of $U_{21} = u_{21} + u_{22}$. The implications of the symmetry condition that corresponds to $U_{12} = U_{21}$ in the continuous time model have been extensively explored by Magill and Scheinkman [1977].

If symmetry is strengthened to separability of U(k,0), that is, $U_{21} = u_{21} + u_{22} = 0$, the stability assumption of Theorem 20 becomes unnecessary.

With this assumption $J(\rho) = \rho(u_{11} + u_{12})$. Since J(1) is negative definite by Assumption I", and in this case $J(\rho) = \rho J(1)$, $J(\rho)$ is negative definite also, which implies regularity by (4). However, we may also prove stability under the assumption of separability. These results are due to Dasgupta [1978].

Consider $(u_{22} + \rho u_{11})^{-1}J(\rho) = I - (1+\rho)(u_{22} + \rho u_{11})^{-1}u_{22} = I + (1+\rho)B^{-1}A$. By a theorem of Arrow ([1974], p. 200) if X is positive quasi-definite and M is symmetric, the real parts of the roots of XM have the same sign distribution as the real parts of the roots of M. In this case $M = -J(\rho)$ and $X = -(u_{22} + \rho u_{11})^{-1}$. Since M is positive definite, the roots of $I + B^{-1}A$ have positive real parts. Indeed, the roots are positive since the roots of $B^{-1}A$ are real as a consequence of the fact that B is definite and A is symmetric (Gantmacher [1960], p. 310). Let r be a root of $I + B^{-1}A$, then r-1 is a root of $B^{-1}A$. But $B^{-1}A$ has negative roots by the same result of Arrow, since it is the product of a positive definite matrix and a negative definite matrix and XM and MX have the same roots. Therefore, $0 < r_i < \frac{1}{1+\rho}$ for all i and (9) has n roots with absolute values less than 1 by Lemma 18. By the result of Araujo and Scheinkman this implies that (k,k) is locally stable, since $\det u_{21} = -\det u_{22} \neq 0$ by Assumption I". We have proved

Theorem 21. *Assume* I". *If* $u_{21} + u_{22} = 0$, *or* U(k,0) *is separable on an optimal stationary path* $k_t = k$, *the optimal stationary state* (k,k) *is locally stable.*

In the continuous time model global stability of an interior optimal stationary path has been proved under the assumption of separability by Scheinkman [1978].

It is an implication of regularity that the utility achieved on the optimal stationary path increases with the discount factor ρ. Indeed, if $k_t = k(\rho')$ is an optimal stationary path and $k(\rho)$ satisfies (1) near ρ', putting $u^*(\rho) = u(k(\rho),k(\rho))$, we have

(13) $\frac{du^*}{d\rho} = (u_1 + u_2)\frac{dk(\rho)}{d\rho}]_{\rho=\rho'}$

Since $u_2 = -\rho u_1$ by (1), (13) implies $\frac{du^*}{d\rho} > 0$ at $\rho = \rho'$ if and only if

$$(1-\rho')u_1 \frac{dk(\rho)}{d\rho} = (1-\rho')q \frac{dk(\rho)}{d\rho} > 0,$$

where the derivatives are evaluated at $\rho = \rho'$. If $0 < \rho' < 1$, the result follows. Following Burmeister and Turnovsky [1972] we may refer to an optimal stationary path that satisfies $\frac{du^*}{d\rho} > 0$ as *non-paradoxical*. Thus we have the result

Theorem 22. *Under Assumption* I", *if an optimal stationary path is regular, it is non-paradoxical.*

The Jacobian matrix of the necessary condition (1) may also be used to study the question of global uniqueness (Brock [1973]). Lemmas 2 and 7, together with Lemma 1 and Theorem 3 imply that (1) is necessary and sufficient for a stationary path that is interior to be optimal when $0 < \rho \leq 1$. Thus the number of solutions to (1) for given ρ and the number of optimal stationary paths for ρ that are interior to D are the same. Recall from the remark following the corollary to Lemma 7 that for $\rho = 1$, an interior optimal stationary path maximizes $u(x,y)$ over $(x,y) \in D$ such that $y-x = 0$. Because the Hessian is negative definite by Assumption I", the maximum is achieved at a unique point. Thus the interior optimal stationary path is unique for $\rho = 1$. The capital stock of this path is also the unique solution of (1) for $\rho = 1$.

Write $G(x,\rho) = u_2(x,x) + \rho u_1(x,x)$ for $(x,x) \in D$ and $0 < \rho \leq 1$. Let C be a convex subset of the diagonal of $E^n \times E^n$, open relative to the diagonal, whose closure $\bar{C}$ lies in the interior of D. We will suppose that for some ρ' such that $0 < \rho' < 1$, all ρ such that $\rho' \leq \rho \leq 1$ have the property that the solutions k of (1) satisfy $(k,k) \in C$. Let C_1 be the projection of C on the first component of the product $E^n \times E^n$. Since C_1 is a convex open subset of Euclidean space,

it is a differentiable manifold and $\bar{C}_1$ is a manifold with boundary. Given ρ, we may use G to define a vector field on $\bar{C}_1$, which for any value of ρ has no zeros on the boundary.

Since G is a continuous function of ρ as ρ varies between ρ' and 1, the vector fields on $\bar{C}_1$ are homotopic to one another. Therefore, the degree of the vector field on the boundary of C_1 is invariant (Milnor [1965], p.). Let $G_x(x,\rho)$ be the derivative G with respect to x. The degree of the vector field on boundary C_1 equals the sum of the signs of det $G_x(k,\rho)$ over all k such that $G(k,\rho) = 0$. But for $\rho = 1$, there is only one such k, and det $G_x(k,1) = (-1)^n$ by Assumption I". Thus the degree of the vector field on boundary $C_1 = (-1)^n$. Assume that the sign of det $G_x(k,\rho)$ does not change over the zeros of the field for some ρ where $\rho' \leq \rho \leq 1$. Then there can be only one zero for G for this ρ, or equivalently only one stationary optimal path interior to D.

At a zero of G, the derivative $G_x(x,\rho)$ is the Jacobian matrix $J(\rho,x)$ of (1). Thus we have proved

Theorem 23. *Let* C *be a convex subset of the diagonal of* $E^n \times E^n$, *open in the relative topology, where* $\bar{C} \subset$ *int* D. *Suppose for* $0 < \rho' < 1$, *all optimal stationary states* k *with* $(k,k) \in$ *int* D, *for all* ρ *with* $\rho' \leq \rho \leq 1$, *satisfy* $(k,k) \in C$. *Then under Assumption I" if the sign of det* $J(\rho,k)$ *is constant over optimal stationary states* k *with* $(k,k) \in$ *int* D, *there is only one such state for this* ρ.

This theorem was first proved, in a neo-classical model, by Benhabib and Nishimura [1979]. It is illustrated in Figure 3.

Sufficient conditions for stability are also useful, as we will see, when parameters of the utility function, other than ρ, are varied. Let $u_t(x,y) = \rho^t u(x,y,\alpha)$, where α is a vector of m parameters of the current utility function. Differentiating (1) totally with respect to α gives

(14) $(u_{21} + u_{22} + \rho u_{11} + \rho u_{12})\frac{dk}{d\alpha} = -u_{2\alpha} - \rho u_{1\alpha}.$

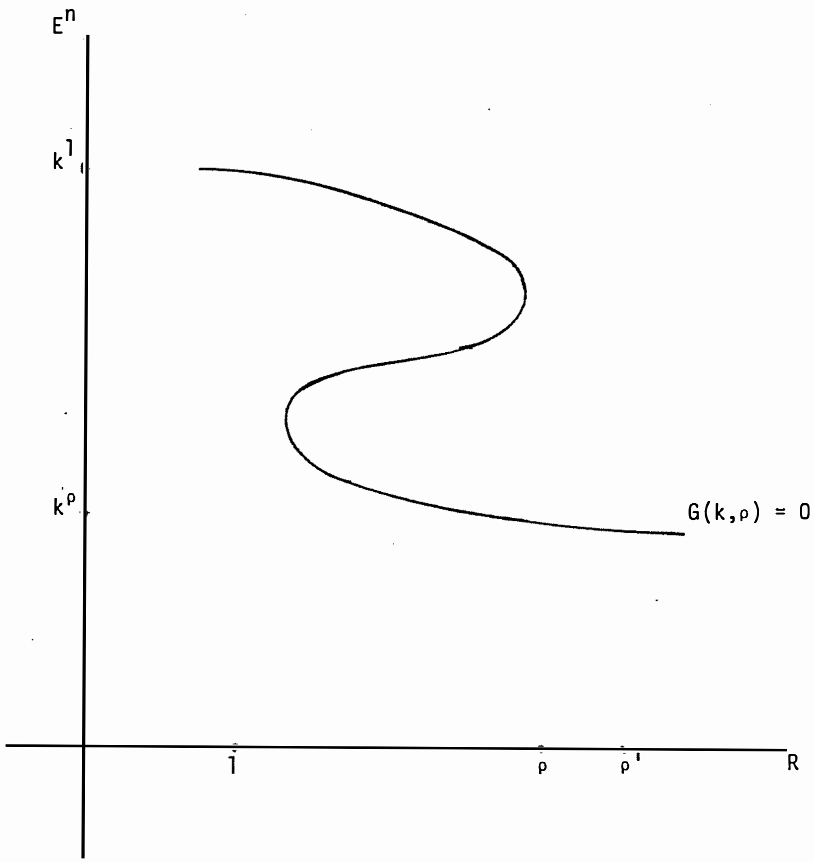

Figure 3

Multiplying (14) on the left by $(\frac{dk}{d\alpha})^T$, an m × n matrix, we have

$$(15)\quad (\frac{dk}{d\alpha})^T(u_{21} + u_{22} + \rho u_{11} + \rho u_{12})\frac{dk}{d\alpha} = -(\frac{dk}{d\alpha})^T(u_{2\alpha} + \rho u_{1\alpha}).$$

A sufficient condition for local stability of the optimal stationary path (k,k) is that $(u_{21} + u_{22} + \rho u_{11} + \rho u_{12})$ be negative quasi-definite at (k,k). Then $(\frac{dk}{d\alpha})^T(u_{2\alpha} + \rho u_{1\alpha})$ is positive quasi-definite. In applications to particular problems, for example, investment of the firm with adjustment cost, this result may be fruitful (Brock [1976]).

Similarly, the sufficient condition (5) for local stability of (k,k), that is, a dominant diagonal for the infinite Jacobian matrix, may be applied to

$$(16)\quad \frac{dk}{d\alpha} = -(I + (u_{22} + \rho u_{11})^{-1} (u_{21} + \rho u_{12}))^{-1} (u_{22} + \rho u_{11})^{-1}\cdot(u_{2\alpha} + \rho u_{1\alpha}).$$

As before, the matrix $-(I + (u_{22} + \rho u_{11})^{-1} (u_{21} + \rho u_{12}))^{-1} (u_{22} + \rho u_{11})^{-1}$ is non-negative when the sign assumptions $(u_{22} + \rho u_{11})^{-1} < 0$, $u_{21} = u_{12}^T \geq 0$, are made. In applications this result may also be useful.

For example, consider a simple model of investment by the firm with adjustment costs, related to the continuous model of Treadway [1971]. Let $\pi(k_t,k_{t+1},\alpha) = f(k_t,k_{t+1}) - \alpha(k_{t+1} - k_t)$, where f represents gross revenue, after maximizing on current spending for variable inputs, and α is a vector of prices for new capital goods. The presence of k_{t+1} as an argument of f is a consequence of the adjustment costs of capital expansion incurred within the firm. The firm's objective is to maximize $\sum_{t=1}^{\infty} \rho^t \pi(k_t,k_{t+1},\alpha)$ given some initial stocks k_0. Prices are formed on competitive markets and are expected to remain constant, while $\rho = \frac{1}{1+r}$, where r is the interest rate, also expected to remain constant.

Differentiating $\pi(k,k,\alpha)$, where $k_t = k$ is an optimal stationary path for the utility function $u_t = \rho^t\pi$, we have $\pi_2 = f_2 - \alpha$ and $\pi_1 = f_1 - \alpha$. Then $\pi_{2\alpha} = -I$ and $\pi_{1\alpha} = -I$. Substituting in the right-hand side of (15), $-(\frac{dk}{d\alpha})^T(u_{2\alpha} + \rho u_{1\alpha})$ becomes $(1+\rho)(\frac{dk}{d\alpha})^T$. If we make the assumption, sufficient for local stability and regularity of (k,k), that $J(\rho) = (\pi_{21} + \pi_{22} + \rho\pi_{11} + \rho\pi_{12})$ is negative quasi-definite, (15) implies that $\frac{dk}{d\alpha}$ is negative quasi-definite, or the equilibrium demand for capital stock is decreasing with respect to prices, in the sense familiar from timeless problems of profit maximization.

We may also apply the assumption of dominant diagonal blocks, with $(\pi_{22} + \rho\pi_{11})^{-1} < 0$ and $\pi_{12} = \pi_{21}^T \geqq 0$, which is also sufficient for local stability of (k,k). Using again that $\pi_{2\alpha} + \rho\pi_{1\alpha} = -(1 + \rho)I$, we infer from (16) that $\frac{dk}{d\alpha} \leqq 0$. This, of course, is what one would expect by analogy to timeless problems with gross substitutes. We may finally note that $\frac{dk}{d\alpha}$ negative quasi-definite also follows from the other assumptions that we used to establish regularity, since the arguments proceeded by way of negative quasi-definite $J(\rho)$. The assumption of symmetry and local stability in Theorem 20, the assumption of separability, and the assumption of U_{21} $(= u_{21} + u_{22})$ negative quasi-definite are cases in point. Another model of investment, used by Lucas [1967], satisfies the separability assumption. It has been studied in this context by Scheinkman [1978].

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